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On the ground state energy for a magnetic Schrödinger operator and the effect of the De Gennes boundary condition

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Abstract

Motivated by the Ginzburg-Landau theory of superconductivity, we estimate in the semi-classical limit the ground state energy of a magnetic Schrödinger operator with De Gennes boundary condition and we study the localization of the ground states. We exhibit cases when the De Gennes boundary condition has strong effects on this localization.

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I. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with regular boundary. Let us consider a cylindrical superconducting sample of cross section Ω. The superconducting properties are described by the minimizers (ψ, A) of the Ginzburg-Landau functional (cf. Refs. 1,2,3) :

$$\mathcal{G}(\psi, A) = \int_\Omega \left\{ |(\nabla - i\sigma\kappa A)\psi|^2 + \sigma^2\kappa^2 |\mathrm{curl\,A} - 1|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 \right\} dx$$
$$+ \int_{\partial\Omega} \tilde{\gamma}\, |\psi(x)|^2 \, d\mu_{|\partial\Omega}(x), \tag{I.1}$$

which is defined for pairs $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$. The parameter κ is a characteristic of the material. A material is said to be of type I if κ is sufficiently small and it is said to be of type II when κ is large. The parameter σ is the intensity of the applied magnetic field which is supposed to be constant and perpendicular to Ω. For a minimizer (ψ, A) of the energy $\mathcal{G}$, the function ψ is called the order parameter and $|\psi|^2$ measures the density of superconducting Cooper electron pairs; the vector field A is called the magnetic potential and $\mathrm{curl}\, A$ is the induced magnetic field. Note that the order parameter ψ satisfies the following boundary condition proposed by De Gennes[1] :

$$\nu \cdot (\nabla - i\sigma\kappa A)\psi + \tilde{\gamma}\psi = 0, \tag{I.2}$$

where ν is the unit outward normal of $\partial\Omega$ and $\tilde{\gamma} \in \mathbb{R}$ is called in the physical literature the De Gennes parameter. Note that the boundary condition (I.2) was initially introduced in the theory of PDE by Robin.

The physicist De Gennes[1] introduced the parameter $\tilde{\gamma}$ in order to model interfaces between superconductors and normal materials. In that context, $\tilde{\gamma}$ is taken to be a non-zero positive constant and $\frac{1}{\tilde{\gamma}}$ (called the extrapolation length) usually measures the penetration of the superconducting Cooper electron pairs in the normal material. The size of $\tilde{\gamma}$ depends on the nature of the material adjacent to the superconductor and it ranges from $\tilde{\gamma} = 0$ (interfaces with insulators) to $\tilde{\gamma} = +\infty$ (interfaces with magnetic and ferromagnetic materials). Experiments show that for superconductors adjacent to ferromagnetic materials, the order parameter ψ vanishes at the boundary[4] and the boundary condition (I.2) is changed to the

Dirichlet boundary condition. Negative values of $\tilde{\gamma}$ were also considered in the physical literature[5]. It is suggested that negative values of $\tilde{\gamma}$ would be useful for modeling the situation when a superconductor is adjacent to another superconductor of higher transition temperature.

Suppose that we have a type II superconductor (i.e. κ is large). The functional $\mathcal{G}$ has a critical point of the type $(0, A)$. Such a critical point is called a normal state. It is then natural to study whether a normal state is a local minimum of $\mathcal{G}$ in the presence of a strong applied magnetic field. The Hessian of $\mathcal{G}$ near a normal state is given by :

$$(\phi, B) \mapsto 2\left[\int_\Omega \left(|(\nabla - i\sigma\kappa A)\phi|^2 - \kappa^2|\phi|^2\right) dx + \int_{\partial\Omega} \tilde{\gamma}\,|\phi|^2 d\mu_{|\partial\Omega}(x) + (\sigma\kappa)^2 \int_\Omega |\mathrm{curl}\, B|^2 dx\right].$$

By defining the change of parameter $h = \frac{1}{\sigma\kappa}$, we have then to study as $h \to 0$ the positivity of the quadratic form :

$$H^1(\Omega) \ni u \mapsto \|(h\nabla - iA)u\|^2_{L^2(\Omega)} + h^2 \int_{\partial\Omega} \tilde{\gamma}\,|u|^2 d\mu_{|\partial\Omega}(x) - (\kappa h)^2 \|u\|^2_{L^2(\Omega)}.$$

The semi-classical limit $h \to 0$ is now equivalent to a large field limit $\sigma \to +\infty$. In order to study the influence of the size of $\tilde{\gamma}$, it seems reasonable to suppose that $\tilde{\gamma}$ is depending on h. Also, due to the possibility of having different materials exterior to Ω together with possible lack of symmetry in the geometry of Ω, it seems also convenient to take $\tilde{\gamma}$ as a function of the boundary. Thus, given a vector field $A \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$, a regular real valued function $\gamma \in C^\infty(\partial\Omega; \mathbb{R})$ and a number $\alpha > 0$, let us define the quadratic form :

$$H^1(\Omega) \ni u \mapsto q^{\alpha,\gamma}_{h,A,\Omega}(u) = \|(h\nabla - iA)u\|^2_{L^2(\Omega)} + h^{1+\alpha} \int_{\partial\Omega} \gamma(x)|u(x)|^2 d\mu_{|\partial\Omega}(x). \tag{I.3}$$

Observing that $q^{\alpha,\gamma}_{h,A,\Omega}$ is semi-bounded, we consider the self-adjoint operator associated to $q^{\alpha,\gamma}_{h,A,\Omega}$ by Friedrich's theorem. This is the magnetic Schrödinger operator $P^{\alpha,\gamma}_{h,A,\Omega}$ with domain $D(P^{\alpha,\gamma}_{h,A,\Omega})$ defined by :

$$\begin{aligned} &P^{\alpha,\gamma}_{h,A,\Omega} = -(h\nabla - iA)^2, \\ &D(P^{\alpha,\gamma}_{h,A,\Omega}) = \{u \in H^2(\Omega); \quad \nu\cdot(h\nabla - iA)u_{|\partial\Omega} + h^\alpha \gamma u_{|\partial\Omega} = 0\}. \end{aligned} \tag{I.4}$$

We denote by $\mu^{(1)}(\alpha,\gamma,h)$ the ground state energy of $P^{\alpha,\gamma}_{h,A,\Omega}$ which is defined using the min-max principle by :

$$\mu^{(1)}(\alpha,\gamma,h) := \inf_{u\in H^1(\Omega),u\neq 0} \frac{q^{\alpha,\gamma}_{h,A,\Omega}(u)}{\|u\|^2_{L^2(\Omega)}}. \tag{I.5}$$

Let us recall also that this eigenvalue problem is gauge invariant.

In the case when $\gamma \equiv 0$ (which corresponds to a superconductor surrounded by the vacuum), a lot of papers are devoted to the estimate in a semiclassical regime of the ground state energy of $P^{\alpha,\gamma}_{h,A,\Omega}$. We would like here to mention the works of Baumann-Phillips-Tang[6], Bernoff-Sternberg[7], del Pino-Felmer-Sternberg[8], Helffer-Mohamed[9], Helffer-Morame[10] and the recent work of Fournais-Helffer[11]. The special case when $\alpha = 1$ and γ is a positive constant was considered by Lu-Pan[12,13]. It was shown that in this case the effect of the De Gennes parameter γ is weak in the sense that the limit $\lim_{h\to 0}\dfrac{\mu^{(1)}(1,\gamma,h)}{h}$ is the same as in the case $\gamma = 0$. This regime is therefore not sufficient to recover all the physically interesting cases considered in Refs. 4,5. It is the object of this paper to establish the results announced in Ref. 14 and to analyze (for all values of α) the influence of the boundary term in (I.3) on the localization of the ground state energy of the operator $P^{\alpha,\gamma}_{h,A,\Omega}$.

Following the technique of Helffer-Morame[10], we have to understand the model case of the half-plane when the magnetic field and the function γ are both constant. Consider the magnetic potential :

$$A_0(x_1,x_2) = \frac{1}{2}(-x_2,x_1), \quad \forall (x_1,x_2)\in \mathbb{R}\times\mathbb{R}_+. \tag{I.6}$$

Notice that $\operatorname{curl} A_0 = 1$. Let us define the function

$$\mathbb{R}\ni\gamma\mapsto\Theta(\gamma),$$

where

$$\Theta(\gamma) := \inf_{u\in H^1_{A_0}(\mathbb{R}\times\mathbb{R}_+),u\neq 0} \frac{\|(\nabla - iA_0)u\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)} + \gamma\int_{\mathbb{R}}|u(x_1,0)|^2dx_1}{\|u\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}}, \tag{I.7}$$

and

$$H^1_{A_0}(\mathbb{R}\times\mathbb{R}_+) = \{u\in L^2(\mathbb{R}\times\mathbb{R}_+);\quad (\nabla - iA_0)u\in L^2(\mathbb{R}\times\mathbb{R}_+)\}. \tag{I.8}$$

Note that $\Theta(\gamma)$ is the bottom of the spectrum of the operator $P^{\alpha,\gamma}_{h,A_0,\Omega}$ with $h = 1$ and $\Omega = \mathbb{R}\times\mathbb{R}_+$. We shall see that $\Theta(\gamma) < 1$ (cf. Theorem II.2). If $\gamma = 0$, we write :

$$\Theta_0 := \Theta(0). \tag{I.9}$$

It is Θ_0 which appears in the analysis for the Neumann problem[6,7,8,9,10,11,12,13]. Actually, we are interested in the bottom of the spectrum of the operator $P^{\alpha,\gamma}_{h,A_0,\mathbb{R}\times\mathbb{R}_+}$ but a scaling argument gives us :

$$\forall h\in\mathbb{R}_+,\ \forall\alpha,\gamma\in\mathbb{R},\quad \inf \mathrm{Sp}\left(P^{\alpha,\gamma}_{h,A_0,\mathbb{R}\times\mathbb{R}_+}\right) = h\Theta(h^{\alpha-1/2}\gamma). \tag{I.10}$$

The semiclassical analysis of the half-plane model depends then on the sign of both $\alpha-\frac{1}{2}$ and γ. We have then to investigate the asymptotic behavior of $\Theta(\gamma)$ when $\gamma\to 0$ and when $\gamma\to\pm\infty$. This will be the object of study in Section II.

Now we state our main results.

Theorem I.1 *Suppose that $\Omega\subset\mathbb{R}^2$ is open, bounded, connected and having a smooth boundary. Suppose moreover that the magnetic field is constant* $\mathrm{curl}\,A = 1$. *Then, for $\alpha > 0$ and $\gamma\in C^\infty(\partial\Omega;\mathbb{R})$, the ground state energy of the operator $P^{\alpha,\gamma}_{h,A,\Omega}$ satisfies :*

$$\mu^{(1)}(\alpha,\gamma,h) = h\Theta(h^{\alpha-1/2}\gamma_0)\left(1+o(1)\right),\quad (h\to 0), \tag{I.11}$$

where $\gamma_0 := \min_{x\in\partial\Omega}\gamma(x)$.

Theorem I.1 gives a first term approximation of $\mu^{(1)}(\alpha,\gamma,h)$. The asymptotics (I.11) is valid without the need to any non-degeneracy hypothesis on the set of minima of γ, and holds for the function γ being constant as well. Let us remark that the asymptotics (I.11) depends strongly on α. In particular, when $\alpha = \frac{1}{2}$, we get :

$$\lim_{h\to 0}\frac{\mu^{(1)}(\alpha,\gamma,h)}{h} = \Theta(\gamma_0) < 1,$$

and if $\gamma_0 = 0$ or if $\alpha > \frac{1}{2}$, then (cf. Proposition II.5) :

$$\lim_{h\to 0}\frac{\mu^{(1)}(\alpha,\gamma,h)}{h} = \Theta_0 < 1.$$

When $\alpha < \frac{1}{2}$, it is the sign of γ_0 that affects the asymptotics. Actually, if $\gamma_0 < 0$ we have (cf. Proposition II.8),

$$\lim_{h\to 0} \frac{\mu^{(1)}(\alpha,\gamma,h)}{h^{2\alpha}} = -\gamma_0^2,$$

and if $\gamma_0 > 0$, we have (cf. (II.46)) :

$$\lim_{h\to 0} \frac{\mu^{(1)}(\alpha,\gamma,h)}{h} = 1$$

which is the same behavior as that for the Dirichlet problem[10]. This last regime ($0 < \alpha < \frac{1}{2}$ and $\gamma_0 > 0$) is in accordance with the physical observations in Ref. 4.

In the next theorem, we give a two-term asymptotics of $\mu^{(1)}(\alpha,\gamma,h)$ when $\alpha \in]\frac{1}{2}, 1[$.

Theorem I.2 *Suppose in addition to the hypotheses of Theorem I.1 that $\frac{1}{2} < \alpha < 1$ and that the function γ is non-constant. Then we have the following asymptotic expansion as h tends to 0 :*

$$\mu^{(1)}(\alpha,\gamma,h) = h\Theta_0 + 6M_3\gamma_0 h^{\alpha+1/2} + \mathcal{O}(h^{\inf(3/2,2\alpha)}), \tag{I.12}$$

where M_3 is a strictly positive universal constant.

The constant M_3 satisfies $\Theta'(0) = 6M_3$ and it will be defined precisely in Section II, see however (II.25) and (II.27). Comparing with the result obtained in Ref. 10, the second term in the two-term asymptotics of $\mu^{(1)}(\alpha,\gamma,h)$ when $\gamma = 0$ is of order $h^{3/2}$, whereas it is of order $h^{\alpha+1/2}$ in the regime considered in Theorem I.2. Let us mention also that in Ref. 11, the authors obtain (when $\gamma = 0$) a complete asymptotic expansion under a generic hypothesis on the scalar curvature of $\partial\Omega$. It seems that a complete asymptotic expansion could be obtained in the regime of Theorem I.2 but under the following generic hypothesis over γ :

- γ has a finite number of minima;
- all the minima of γ are non-degenerate.

We leave this point hoping to analyze it in a future work.

Next we turn to the question of the localization of the ground states. Let $u_{\alpha,\gamma,h}$ be a ground state of the operator $P^{\alpha,\gamma}_{h,A,\Omega}$. We say that $u_{\alpha,\gamma,h}$ is exponentially localized as h tends to 0 near a closed set $\mathcal{B}$ in $\overline{\Omega}$ if there exists $\beta > 0$, and for each neighborhood $\mathcal{V}$ of $\mathcal{B}$, there exist positive constants h_0, δ and C such that :

$$\|u_{\alpha,\gamma,h}\|_{L^2(\Omega\setminus\mathcal{V})} \leq C\exp\left(-\frac{\delta}{h^\beta}\right)\|u_{\alpha,\gamma,h}\|_{L^2(\Omega)}, \quad \forall h \in]0,h_0]. \tag{I.13}$$

In the next theorem we describe some effect of γ on the localization of the ground states of the operator $P^{\alpha,\gamma}_{h,A,\Omega}$.

Theorem I.3 *Under the hypotheses of Theorem I.1, if $\gamma_0 \leq 0$ or $\frac{1}{2} \leq \alpha < 1$, a ground state of the operator $P^{\alpha,\gamma}_{h,A,\Omega}$ is exponentially localized as h tends to 0 near the boundary points where γ is minimum.*
More precisely, (I.13) is satisfied with $\beta = 1-\alpha$ if $\gamma_0 < 0$, $\beta = \frac{1-\alpha}{2}$ if $\frac{1}{2} < \alpha < 1$, and $\beta = 1/2$ otherwise.

In the special case $\alpha = 1$, the scalar curvature κ_{r} and the function γ affects the asymptotic expansion of the ground state energy to the same order.

Theorem I.4 *Suppose in addition to the hypotheses of Theorem I.1 that $\alpha = 1$. Then we have the following asymptotic expansion as h tends to 0 :*

$$\mu^{(1)}(\alpha,\gamma,h) = h\Theta_0 - 2M_3(\kappa_{\mathrm{r}} - 3\gamma)_{\max}h^{3/2} + \mathcal{O}(h^{13/8}), \tag{I.14}$$

and a ground state $u_{\alpha,\gamma,h}$ of the operator $P^{1,\gamma}_{h,A,\Omega}$ is localized near the boundary points where the function $\kappa_{\mathrm{r}} - 3\gamma$ is maximal.
More precisely, (I.13) is satisfied with $\beta = 1/4$.

If γ is constant, the remainder in (I.14) is better and of order $\mathcal{O}(h^{5/3})$. When $\gamma \equiv 0$ we recover in the above theorem the result of Helffer-Morame[10]. Let us mention that the expansion (I.14) is announced by Pan[25] in the particular case when γ is a positive constant. As in Ref. 11, we believe that an asymptotic expansion with higher terms could be obtained

under a generic hypothesis on the function $\kappa_{\mathrm{r}} - 3\gamma$.

In the next theorem, we study the case when the function γ is constant and we find that only the scalar curvature plays a role.

Theorem I.5 *Suppose in addition to the hypotheses of Theorem I.1 that the function γ is constant and that $\alpha \geq \frac{1}{2}$. There exists a constant $M_3(\alpha,\gamma) > 0$ such that we have the following asymptotic expansion as h tends to 0 :*

$$\mu^{(1)}(\alpha,\gamma,h) = h\Theta(h^{\alpha-1/2}\gamma) - 2M_3(\alpha,\gamma)(\kappa_{\mathrm{r}})_{\max}h^{3/2} + o(h^{3/2}). \tag{I.15}$$

Moreover, a ground state of the operator $P^{\alpha,\gamma}_{h,A,\Omega}$ is localized as h tends to 0 near the boundary points where the scalar curvature is maximal, and (I.13) is satisfied with $\beta = 1/4$.

When $\alpha > \frac{1}{2}$, the constant $M_3(\alpha,\gamma)$ is equal to the universal constant M_3. When $\alpha = \frac{1}{2}$, we have $M_3(\frac{1}{2},\gamma) = M_3(\gamma)$, where the constant $M_3(\gamma)$ will be defined in Section II (cf. (II.24)).

This paper is organized in the following way. In Section II, we link the analysis of the half-plane model operator to that of a one dimensional operator. We get in particular the existence of a number $\xi(\gamma) > 0$ such that $\Theta(\gamma)$ is the lowest eigenvalue of the operator $-\partial_t^2 + (t-\xi(\gamma))^2$. Let φ_γ be an eigenfunction associated to $\Theta(\gamma)$. We establish the regularity of $\Theta(\gamma)$ and φ_γ as functions of γ, the asymptotic behavior of $\Theta(\gamma)$ as $\gamma \to \pm\infty$, and uniform estimates with respect to γ describing the exponential decay of φ_γ at infinity.

In Section III, we use the eigenfunction φ_γ to construct a test function inspired by Refs. 7,10 and we obtain an upper bound for $\mu^{(1)}(\alpha,\gamma,h)$. We then carry out a similar analysis to that in Ref. 10 and we use the results of Section II to prove Theorem I.1.

In Section IV, we show how to get the localization of the ground states using Agmon's technique[15]. Finally, in Section V, the analysis of a one-dimensional family of operators on a weighted L^2-space appears (cf. (V.21)). It is the same family of operators appearing in Ref. 10 (Section 11) but with a different boundary condition this time. This analysis permits

us to derive two-term asymptotics of the ground state energy showing the influence of the scalar curvature. We finish then the proofs of Theorems I.2, I.3, I.4 and I.5.

II. THE MODEL OPERATOR

Given $\gamma \in \mathbb{R}$, let us consider the quadratic form :

$$H^1_{A_0}(\mathbb{R}\times\mathbb{R}_+) \ni u \mapsto q[\gamma](u) = \|(\nabla - iA_0)u\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)} + \gamma\int_{\mathbb{R}} |u(x_1,0)|^2 dx_1. \tag{II.1}$$

The magnetic potential A_0 and the form domain $H^1_{A_0}(\mathbb{R}\times\mathbb{R}_+)$ are defined respectively in (I.6) and (I.8). Observing that the quadratic form $q[\gamma]$ is bounded from below, we can associate to $q[\gamma]$, by taking the Friedrichs extension, a unique self-adjoint operator $P[\gamma]$ on $L^2(\mathbb{R}\times\mathbb{R}_+)$. The min-max principle gives that the bottom of the spectrum of $P[\gamma]$ is equal to $\Theta(\gamma)$ (cf. (I.7)).

A. Link with a one dimensional operator

By a change of gauge and a partial Fourier transformation with respect to the first variable, we obtain that the spectral analysis of the operator $P[\gamma]$ will be deduced from that of the ξ-family of one dimensional operators :

$$H[\gamma,\xi] = -\frac{d^2}{dt^2} + (t-\xi)^2, \tag{II.2}$$

with domain

$$D(H[\gamma,\xi]) = \{u \in B^2(\mathbb{R}_+);\quad u'(0) = \gamma u(0)\}, \tag{II.3}$$

where, for a given integer k, the space $B^k(\mathbb{R}_+)$ is defined by :

$$B^k(\mathbb{R}_+) = \{u \in H^k(\mathbb{R}_+);\quad t^k u \in L^2(\mathbb{R}_+)\}. \tag{II.4}$$

Note that the operator $H[\gamma,\xi]$ has compact resolvent and hence the spectrum is discrete. We denote by $\mu^{(1)}(\gamma,\xi)$ the first eigenvalue of $H[\gamma,\xi]$. The min-max principle gives :

$$\mu^{(1)}(\gamma,\xi) = \inf_{u\in B^1(\mathbb{R}_+),u\neq 0} \frac{q[\gamma,\xi](u)}{\|u\|^2_{L^2(\mathbb{R}_+)}},$$

where $q[\gamma,\xi]$ is the quadratic form associated to $H[\gamma,\xi]$:

$$q[\gamma,\xi](u) = \int_{\mathbb{R}_+} \left(|u'(t)|^2 + |(t-\xi)u(t)|^2\right) dt + \gamma|u(0)|^2. \tag{II.5}$$

A spectral analysis using the separation of variables (cf. Ref. 16) gives us :

$$\Theta(\gamma) = \inf_{\xi\in\mathbb{R}} \mu^{(1)}(\gamma,\xi). \tag{II.6}$$

In the following lemma, we collect some useful estimates of $\mu^{(1)}(\gamma,\xi)$.

Lemma II.1 *Given $\epsilon \in]0,1[$, we have,*

$$\mu^{(1)}(\gamma,\xi) \geq (1-\epsilon)\mu^{(1)}(0,\xi) - \frac{(\gamma_-)^2}{\epsilon}, \quad \forall \gamma,\xi \in \mathbb{R}, \tag{II.7}$$

where $\gamma_- = \max(-\gamma, 0)$.

Moreover, given $\gamma \in \mathbb{R}$, we have :

$$\lim_{\xi\to-\infty} \mu^{(1)}(\gamma,\xi) = +\infty, \quad \lim_{\xi\to+\infty} \mu^{(1)}(\gamma,\xi) = 1. \tag{II.8}$$

Proof. Using the density of $C_0^\infty(\overline{\mathbb{R}_+})$ in $H^1(\mathbb{R}_+)$, we get for any $u \in H^1(\mathbb{R}_+)$:

$$|u(0)|^2 = -2\int_0^\infty u(t)u'(t)dt. \tag{II.9}$$

By the Cauchy-Schwarz inequality, we get for any $\alpha > 0$:

$$|u(0)|^2 \leq \alpha\|u\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{\alpha}\|u'\|^2_{L^2(\mathbb{R}_+)}.$$

Taking $\alpha = \frac{\epsilon}{\gamma}$ (with $\gamma < 0$), we get :

$$q[\gamma,\xi](u) \geq (1-\epsilon)q[0,\xi](u) - \frac{(\gamma_-)^2}{\epsilon}\|u\|^2_{L^2(\mathbb{R}_+)}, \quad \forall u \in B^1(\mathbb{R}_+). \tag{II.10}$$

The min-max principle now gives (II.7).

Notice that (II.8) is valid for $\gamma = 0$ (Ref. 10). So the limit as $\xi \to -\infty$ in (II.8) is now a consequence of the estimate (II.7). For the reader's convenience, let us give for non-zero γ a proof for the limit as $\xi \to +\infty$ in (II.8). Let us denote by $\mu^D(\xi)$ the first eigenvalue of the Dirichlet realization of the harmonic oscillator $-\partial_t^2 + (t-\xi)^2$ on $\mathbb{R}_+$. We have by the min-max principle :

$$\mu^{(1)}(0,\xi) + \gamma|\varphi_{\gamma,\xi}(0)|^2 \leq \mu^{(1)}(\gamma,\xi) \leq \mu^D(\xi), \tag{II.11}$$

where $\varphi_{\gamma,\xi}$ is the L^2-normalized eigenfunction associated to $\mu^{(1)}(\gamma,\xi)$. Let us notice also that[10,16] $\lim_{\xi\to+\infty}\mu^D(\xi)=1$. So, if we know that $\lim_{\xi\to+\infty}|\varphi_{\gamma,\xi}(0)|^2=0$, then (II.11) is sufficient to deduce the limit as $\xi\to+\infty$ in (II.8). Thus, it remains for us to prove the following claim :

$$\begin{aligned}&\text{Given } \epsilon\in]0,1[\text{ and } \gamma\in\mathbb{R}, \text{ there exists a constant } C>0 \text{ such that,}\\ &\forall\xi\in[C,+\infty[,\quad |\varphi_{\gamma,\xi}(0)|^2\leq Ce^{-\epsilon\xi/2}.\end{aligned}\tag{II.12}$$

Let us mention that the decay in (II.12) is not optimal[17,18]. We prove (II.12) using Agmon type estimates[15]. Let Φ be a regular function with compact support. An integration by parts gives the following identity :

$$q[\gamma,\xi]\left(e^{\Phi}\varphi_{\gamma,\xi}\right)=\mu^{(1)}(\gamma,\xi)\left\|e^{\Phi}\varphi_{\gamma,\xi}\right\|^2_{L^2(\mathbb{R}_+)}+\left\|\Phi' e^{\Phi}\varphi_{\gamma,\xi}\right\|^2_{L^2(\mathbb{R}_+)}.\tag{II.13}$$

Using the estimate (II.10) (with $\epsilon=1/2$) together with the fact that $\mu^D(\xi)$ is bounded for $\xi\in\mathbb{R}_+$, we can rewrite (II.13) in the form :

$$\frac{1}{2}\left(\left\|\left(e^{\Phi}\varphi_{\gamma,\xi}\right)'\right\|^2_{L^2(\mathbb{R}_+)}+\left\|(t-\xi)e^{\Phi}\varphi_{\gamma,\xi}\right\|^2_{L^2(\mathbb{R}_+)}\right)\leq\tilde{C}\left\|e^{\Phi}\varphi_{\gamma,\xi}\right\|^2_{L^2(\mathbb{R}_+)}+\left\|\Phi' e^{\Phi}\varphi_{\gamma,\xi}\right\|^2_{L^2(\mathbb{R}_+)},\tag{II.14}$$

for some constant $\tilde{C}>0$. We choose now Φ as :

$$\Phi(t):=\begin{cases}\epsilon\xi\frac{(t-1)^2}{2} & if\ \ 0\leq t\leq 1,\\ 0 & if\ \ t\geq 1.\end{cases}$$

Under this choice of Φ, we can get a sufficiently large constant $C>0$ such that, for $\xi\in[C,+\infty[$, we can rewrite (II.14) in the form :

$$\left\|e^{\Phi}\varphi_{\gamma,\xi}\right\|_{H^1(\Omega)}\leq C.$$

Using the Sobolev imbedding $H^1(\mathbb{R}_+)\hookrightarrow L^\infty(\mathbb{R}_+)$, this last estimate is sufficient to deduce (II.12). $\square$

Following the analysis of Dauge-Helffer[19], we have now the following result.

Theorem II.2 *For each $\gamma \in \mathbb{R}$, $\Theta(\gamma) < 1$ and the function $\mathbb{R} \ni \xi \mapsto \mu^{(1)}(\gamma,\xi)$ attains its minimum at a unique positive point $\xi(\gamma)$ that satisfies :*

$$\xi(\gamma)^2 = \Theta(\gamma) + \gamma^2. \tag{II.15}$$

Proof. Let us notice that by Kato's theory (Ref. 20), the maps

$$\xi \mapsto \mu^{(1)}(\gamma,\xi), \quad \xi \mapsto \varphi_{\gamma,\xi} \in L^2(\mathbb{R}_+)$$

are analytic. Here we recall that $\varphi_{\gamma,\xi}$ is the unique strictly positive and L^2-normalized eigenfunction associated to $\mu^{(1)}(\gamma,\xi)$. Let us consider $\tau > 0$. Note that,

$$\mu^{(1)}(\gamma,\xi+\tau)\varphi_{\gamma,\xi+\tau}(t+\tau) = H[\gamma,\xi]\left(\varphi_{\gamma,\xi+\tau}(t+\tau)\right), \quad \forall t \in \mathbb{R}_+.$$

Taking the scalar product with $\varphi_{\gamma,\xi}$ and then integrating by parts, we get :

$$\begin{aligned}&\left(\mu^{(1)}(\gamma,\xi+\tau) - \mu^{(1)}(\gamma,\xi)\right)\int_{\mathbb{R}_+} \varphi_{\gamma,\xi+\tau}(t+\tau)\varphi_{\gamma,\xi}(t)\,dt \\ &= \varphi'_{\gamma,\xi+\tau}(\tau)\varphi_{\gamma,\xi}(0) - \gamma\varphi_{\gamma,\xi+\tau}(\tau)\varphi_{\gamma,\xi}(0).\end{aligned} \tag{II.16}$$

Recall that we have the boundary conditions :

$$\varphi'_{\gamma,\xi+\tau}(0) = \gamma\varphi_{\gamma,\xi+\tau}(0), \quad \varphi'_{\gamma,\xi}(0) = \gamma\varphi_{\gamma,\xi}(0).$$

Then we can rewrite (II.16) as :

$$\begin{aligned}&\frac{\mu^{(1)}(\gamma,\xi+\tau) - \mu^{(1)}(\gamma,\xi)}{\tau}\int_{\mathbb{R}_+} \varphi_{\gamma,\xi+\tau}(t+\tau)\varphi_{\gamma,\xi}(t)\,dt \\ &= \left[\frac{\varphi'_{\gamma,\xi+\tau}(\tau) - \varphi'_{\gamma,\xi+\tau}(0)}{\tau} - \gamma\frac{\varphi_{\gamma,\xi+\tau}(\tau) - \varphi_{\gamma,\xi+\tau}(0)}{\tau}\right]\cdot\varphi_{\gamma,\xi}(0).\end{aligned}$$

By taking the limit as $\tau \to 0$, we get :

$$\partial_\xi\mu^{(1)}(\gamma,\xi) = \left(\varphi''_{\gamma,\xi}(0) - \gamma\varphi'_{\gamma,\xi}(0)\right)\varphi_{\gamma,\xi}(0).$$

Finally, we make the substitutions :

$$\varphi''_{\gamma,\xi}(0) = \left(\xi^2 - \mu^{(1)}(\gamma,\xi)\right)\varphi_{\gamma,\xi}(0), \quad \varphi'_{\gamma,\xi}(0) = \gamma\varphi_{\gamma,\xi}(0),$$

and we get the following formula,

$$\partial_\xi \mu^{(1)}(\gamma,\xi) = \left(\xi^2 - \mu^{(1)}(\gamma,\xi) - \gamma^2\right) |\varphi_{\gamma,\xi}(0)|^2, \tag{II.17}$$

called usually the F-formula (cf. Refs. 19,21). Using (II.7) and (II.8), we get :

$$\partial_\xi \mu^{(1)}(\gamma,\xi)_{|\xi=0} < 0, \quad \partial_\xi \mu^{(1)}(\gamma,\xi)_{|\xi=\eta} > 0,$$

for a sufficiently large $\eta > 0$. This gives the existence of a positive critical point of $\mu^{(1)}(\gamma,\xi)$. Let us notice now that for any critical point ξ_c of $\mu^{(1)}(\gamma,\xi)$, we have :

$$\partial_\xi^2 \mu^{(1)}(\gamma,\xi)_{|\xi=\xi_c} = 2\xi_c |\varphi_{\gamma,\xi}(0)|^2.$$

This shows that any negative critical point is a global maximum and any positive critical point is a global minimum of $\mu^{(1)}(\gamma,\xi)$. Coming back to (II.8), $\lim\limits_{\xi\to-\infty} \mu^{(1)}(\gamma,\xi) = +\infty$, and thus there does not exist any negative critical points. Therefore, the minimum of $\xi \mapsto \mu^{(1)}(\gamma,\xi)$ is attained at a unique point $\xi(\gamma) > 0$ and the function $\xi \mapsto \mu^{(1)}(\gamma,\xi)$ is strictly increasing on $[\xi(\gamma), +\infty[$. This proves in particular (recalling (II.6)) :

$$\Theta(\gamma) = \mu^{(1)}(\gamma, \xi(\gamma)) < 1.$$

$\square$

In the sequel, we denote by φ_γ the unique strictly positive and L^2-normalized eigenfunction associated to the eigenvalue $\Theta(\gamma)$, and by $H[\gamma]$ the operator $H[\gamma,\xi(\gamma)]$:

$$\varphi_\gamma = \varphi_{\gamma,\xi(\gamma)}, \quad H[\gamma] = H[\gamma,\xi(\gamma)]. \tag{II.18}$$

In the next lemma, we collect various useful relations satisfied by the eigenfunction φ_γ. These relations are similar to those given in Appendix A of Ref. 10.

Lemma II.3 *For each $\gamma \in \mathbb{R}$, the following relations hold :*

$$\int_{\mathbb{R}_+} (t - \xi(\gamma)) |\varphi_\gamma(t)|^2 dt = 0, \tag{II.19}$$

$$\int_{\mathbb{R}_+} (t-\xi(\gamma))^2 |\varphi_\gamma(t)|^2 dt = \frac{\Theta(\gamma)}{2} - \frac{\gamma}{4} |\varphi_\gamma(0)|^2, \tag{II.20}$$

$$\int_{\mathbb{R}_+} (t-\xi(\gamma))^3 |\varphi_\gamma(t)|^2 dt = \frac{1}{6}\left[1-2(\gamma\xi(\gamma))^2\right] |\varphi_\gamma(0)|^2. \tag{II.21}$$

Proof. We follow the calculations done in Bernoff-Sternberg[7]. Let us consider the differential operator :

$$L = -\partial_t^2 + (t-\xi(\gamma))^2 - \Theta(\gamma).$$

Note that for any polynomial p, we have the following identity :

$$L(2p\varphi_\gamma - p'\varphi_\gamma) = \left(p^{(3)} - 4\left[(t-\xi(\gamma) - \Theta(\gamma)\right] p' - 4(t-\xi(\gamma))p\right) \varphi_\gamma. \tag{II.22}$$

Let $v = 2p\varphi_\gamma' - p'\varphi_\gamma$. Integrating by parts we obtain :

$$\int_0^{+\infty} \varphi_\gamma(t)(Lv)(t)\, dt = (v'(0) - \gamma v(0))\varphi_\gamma(0). \tag{II.23}$$

Taking $p = 1$, we get :

$$-4\int_0^{+\infty} (t-\xi(\gamma))|\varphi_\gamma(t)|^2 dt = 2\left(\xi(\gamma)^2 - \gamma^2 - \Theta(\gamma)\right) |\varphi_\gamma(0)|^2.$$

Recalling (II.15), the above formula proves (II.19).

We prove (II.20) by taking $p = (t-\xi(\gamma))$. To prove (II.21), we take $p = (t-\xi(\gamma))^2$. Note that we have in this case :

$$v'(0) - \gamma v(0) = 2\left(2(\gamma\xi(\gamma))^2 - 1\right) \varphi_\gamma(0).$$

We get now from (II.22) and (II.23) :

$$-12\int_0^{+\infty} (t-\xi(\gamma))^3 |\varphi_\gamma(t)|^2 dt = 2\left(2(\gamma\xi(\gamma))^2 - 1\right) |\varphi_\gamma(0)|^2.$$

This proves (II.21). $\square$

For $\gamma \in \mathbb{R}$, let us define the parameter :

$$M_3(\gamma) = \frac{1}{6}\left(1 + (\gamma\xi(\gamma))^2\right) |\varphi_\gamma(0)|^2, \tag{II.24}$$

and when $\gamma = 0$, we write $M_3 := M_3(0)$. Note that (II.21) gives :

$$M_3 = \int_{\mathbb{R}_+} (t - \xi_0)^3 |\varphi_0(t)|^2 dt, \tag{II.25}$$

where $\xi_0 := \xi(0)$. The constant M_3 is the universal constant appearing in Theorems I.2, I.4, and the parameter $M_3(\gamma)$ appears as $M_3(\frac{1}{2}, \gamma)$ in Theorem I.5.

B. Regularity

We discuss now the regularity of the functions $\gamma \to \Theta(\gamma) \in \mathbb{R}$ and $\gamma \mapsto \varphi_\gamma \in L^2(\mathbb{R}_+)$. It seems for us that Kato's theory (cf. Ref. 20) do not apply in this context at least for the reason that we do not know a priori whether the expression of the operator

$$H[\gamma] = -\frac{d^2}{dt^2} + (t - \xi(\gamma))^2$$

depends analytically on γ. Inspired by Bonnaillie[22], we use a modification of Grushin's method[23] and we get the following proposition.

Proposition II.4 *The functions $\mathbb{R} \ni \gamma \mapsto \Theta(\gamma) \in \mathbb{R}$ and $\mathbb{R} \ni \gamma \mapsto \varphi_\gamma \in L^2(\mathbb{R}_+)$ are C^∞. Moreover, the function $\mathbb{R} \ni \gamma \mapsto \varphi_\gamma \in L^\infty(\mathbb{R}_+)$ is locally Lipschitz.*

The specific difficulty in proving Proposition II.4 comes from the fact that both the expression and the domain of the operator $H[\gamma]$ depend on γ. To work with an operator with a fixed domain, we consider a cut-off χ that is equal to 1 on $[0, 1]$ and we apply the invertible transformation $\varphi \mapsto \tilde{\varphi} = e^{-\gamma t \chi(t)} \varphi$ that transforms the boundary condition $\varphi'(0) = \gamma\varphi(0)$ to the usual Neumann boundary condition $\tilde{\varphi}'(0) = 0$ and leaves the spectrum invariant (cf. Proof of Proposition II.7).

In the next proposition we determine $\Theta'(\gamma)$. This is a first step in the proof of Proposition II.4.

Proposition II.5 *The function $\gamma \mapsto \Theta(\gamma)$ is of class C^1 and satisfies :*

$$\Theta'(\gamma) = |\varphi_\gamma(0)|^2. \tag{II.26}$$

In particular, we have :

$$\Theta'(0) = 6M_3. \tag{II.27}$$

Remark II.6 *Using Formula (II.15) we get also that the function $\gamma \mapsto \xi(\gamma)$ is of class C^1.*

Proof of Proposition II.5. Let τ be a real number. We shall define the following trial function :

$$u = e^{\tau t}(\varphi_\gamma + \tau u_1),$$

where

$$u_1 = (H[\gamma] - \Theta(\gamma))^{-1} \left\{ |\varphi_\gamma(0)|^2 \varphi_\gamma + 2\varphi_\gamma' + 2(\xi(\gamma+\tau) - \xi(\gamma))(t - \xi(\gamma))\varphi_\gamma \right\}.$$

By standard Fredholm theory, the operator $(H[\gamma] - \Theta(\gamma))^{-1}$ is defined on the orthogonal space of φ_γ and has values in $D(H[\gamma])$. Hence, the function u_1 is well defined, thanks to (II.19), and the function u satisfies the boundary condition $u'(0) = (\gamma + \tau)u(0)$. When τ is sufficiently small, it is a result of the exponential decay of φ_γ at $+\infty$ (cf. Propositions II.9 and II.10) and standard elliptic estimates that $u \in B^2(\mathbb{R}_+)$. Therefore, $u \in D(H[\gamma + \tau])$, and we have :

$$H[\gamma+\tau](u) = e^{\tau t} \left(-\partial_t^2 + (t - \xi(\gamma+\tau))^2 - 2\tau\partial_t - \tau^2 \right) (\varphi_\gamma + \tau u_1). \tag{II.28}$$

Using the decomposition :

$$H[\gamma+\tau] = H[\gamma] - 2(\xi(\gamma+\tau) - \xi(\gamma))(t - \xi(\gamma)) + (\xi(\gamma+\tau) - \xi(\gamma))^2,$$

we can rewrite (II.28) as :

$$\begin{aligned} &\left(H[\gamma+\tau] - \Theta(\gamma) - |\varphi_\gamma(0)|^2 \tau \right)(u) \\ &= \tau^2 e^{\tau t} \left(\Theta(\gamma) + |\varphi_\gamma(0)|^2 \right) u_1 + \left(\xi(\gamma+\tau) - \xi(\gamma) \right)^2 u. \end{aligned} \tag{II.29}$$

We make the following claim :

$$\forall\gamma\in\mathbb{R},\quad \exists\, C>0,\quad \forall\tau\in[-1,1],\quad |\xi(\gamma+\tau)-\xi(\gamma)|\leq C|\tau|. \tag{II.30}$$

Therefore, thanks to (II.29) and (II.30), there exist constants $\tilde{C},\tau_0>0$ such that, for all $\tau\in[-\tau_0,\tau_0]$, we have :

$$\left\|\left(H[\gamma+\tau]-\Theta(\gamma)-|\varphi_\gamma(0)|^2\tau\right)u\right\|_{L^2(\mathbb{R}_+)}\leq \tilde{C}\tau^2\|u\|_{L^2(\mathbb{R}_+)}.$$

We get now by the spectral theorem the existence of an eigenvalue $\tilde{\Theta}(\gamma+\tau)$ of the operator $H[\gamma+\tau]$ that satisfies the following estimate :

$$|\tilde{\Theta}(\gamma+\tau)-\Theta(\gamma)-|\varphi_\gamma(0)|^2\tau|\leq\tilde{C}\tau^2,\quad \forall\tau\in[-\tau_0,\tau_0]. \tag{II.31}$$

We make now another claim :

$$\forall\gamma\in\mathbb{R},\quad \exists\, C_1>0,\quad \forall\tau\in[-1,1],\quad \left|\mu^{(2)}(\gamma+\tau,\xi(\gamma+\tau))-\mu^{(2)}(\gamma,\xi(\gamma))\right|\leq C|\tau|, \tag{II.32}$$

where for $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}$, $\mu^{(2)}(\eta,\xi)$ denotes the second eigenvalue of the operator $H[\eta,\xi]$. Under the above claim, the estimate (II.31) gives :

$$\tilde{\Theta}(\gamma+\tau)=\Theta(\gamma+\tau),\quad \forall\tau\in[-\tau_0,\tau_0].$$

Consequently, we get that $\Theta(\gamma)$ is differentiable and satisfies formula (II.26). We make now a final claim :

$$\text{The function } \gamma\mapsto|\varphi_\gamma(0)|^2 \text{ is locally Lipschitz.} \tag{II.33}$$

To achieve the proof of the theorem, we only need to prove (II.30), (II.32) and (II.33).

Proof of (II.30).

As we have the formula (II.15), it is sufficient to prove :

$$\forall\gamma\in\mathbb{R},\quad \exists\, C>0,\quad \forall\tau\in[-1,1],\quad |\Theta(\gamma+\tau)-\Theta(\gamma)|\leq C|\tau|. \tag{II.34}$$

The min-max principle gives :

$$\mu^{(1)}(\gamma,\xi)+\tau|\varphi_{\gamma+\tau,\xi}(0)|^2\leq\mu^{(1)}(\gamma+\tau,\xi)\leq\mu^{(1)}(\gamma,\xi)+\tau|\varphi_{\gamma,\xi}(0)|^2,\quad \forall\xi\in\mathbb{R}. \tag{II.35}$$

Thus, given an eigenfunction φ of $H[\gamma,\xi]$, we need to estimate $|\varphi(0)|^2$. Let $u \in D(H[\gamma])$. Using (II.9) we get :

$$|u(0)|^2 \leq 2\|u\|_{L^2(\mathbb{R}_+)}\|u'\|_{L^2(\mathbb{R}_+)}. \tag{II.36}$$

We use now (II.10) (with $\epsilon = 1/2$) to obtain :

$$\|u'\|^2_{L^2(\mathbb{R}_+)} \leq 2q[\gamma,\xi](u) + (\gamma_-)^2\|u\|^2_{L^2(\mathbb{R}_+)}. \tag{II.37}$$

Combining (II.36) and (II.37), we get after an integration by parts,

$$|u(0)|^2 \leq 2\,\|H[\gamma,\xi]u\|_{L^2(\mathbb{R}_+)}\,\|u\|_{L^2(\mathbb{R}_+)} + (\gamma_-)^2\|u\|^2_{L^2(\mathbb{R}_+)}, \quad \forall u \in D(H[\gamma]). \tag{II.38}$$

Let $M := \sup_{\tau\in[-1,1]} \xi(\gamma+\tau)$. Let us show that $M < +\infty$. Actually, the min-max principle gives :

$$\mu^{(1)}(\gamma-1,\xi) \leq \mu^{(1)}(\gamma+\tau,\xi) \leq \mu^{(1)}(\gamma+1,\xi), \quad \forall\tau\in[-1,1], \quad \forall\xi\in\mathbb{R}.$$

Recalling (II.6), we obtain $\Theta(\gamma-1) \leq \sup_{\tau\in[-1,1]} \Theta(\gamma+\tau) \leq \Theta(\gamma+1)$. Formula (II.15) now gives $M < +\infty$. Therefore, (II.38) gives :

$$|\varphi_{\gamma,\xi}(0)|^2 \leq C, \quad |\varphi_{\gamma+\tau,\xi}(0)|^2 \leq C, \quad \forall\tau\in[-1,1], \quad \forall\xi\in[-M,M],$$

for some constant $C > 0$. Consequently (II.35) yields the estimate :

$$\mu^{(1)}(\gamma,\xi) - C\tau_- \leq \mu^{(1)}(\gamma+\tau,\xi) \leq \mu^{(1)}(\gamma,\xi) + C\tau_+, \quad \forall\xi\in[-M,M].$$

Minimizing with respect to ξ, we get (II.34), thanks to Theorem II.2.

Proof of (II.32).

Let $u \in B^1(\mathbb{R}_+)$. We shall compare $q[\gamma+\tau,\xi(\gamma+\tau)](u)$ and $q[\gamma,\xi(\gamma)](u)$. In fact, we have :

$$\begin{aligned} q[\gamma+\tau,\xi(\gamma+\tau)][u] \;=\; & q[\gamma,\xi(\gamma)](u) - 2(\xi(\gamma+\tau)-\xi(\gamma))\int_0^{+\infty}(t-\xi(\gamma))|u(t)|^2dt \\ & +(\xi(\gamma+\tau)-\xi(\gamma))^2\int_0^{+\infty}|u(t)|^2dt + \tau|u(0)|^2, \end{aligned}$$

where, combining (II.36) and (II.37),

$$|u(0)|^2 \leq 2q[\gamma,\xi](u) + (\gamma_-)^2\|u\|^2_{L^2(\mathbb{R}_+)}.$$

The Cauchy-Schwarz inequality gives :

$$2\left|\int_0^{+\infty}(t-\xi(\gamma))|u(t)|^2dt\right| \leq \int_0^{+\infty}|(t-\xi(\gamma))u(t)|^2\,dt + \int_0^{+\infty}|u(t)|^2dt.$$

Using (II.30), we get a constant $C>0$ such that, for all $\tau\in[-1,1]$, we have :

$$\begin{aligned}(1-C\tau_-)q[\gamma,\xi(\gamma)](u) - C\tau_-\|u\|^2_{L^2(\mathbb{R}_+)} &\leq q[\gamma,\xi(\gamma+\tau)](u)\\ &\leq (1+C\tau_+)q[\gamma,\xi(\gamma)](u)+C\tau_+\|u\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

The min-max principle proves now the claim.

Proof of (II.33).

Let $u=\varphi_\gamma(t)-e^{-\tau t}\varphi_{\gamma+\tau}(t)$. It is sufficient to prove that :

$$|u(0)|^2\leq C|\tau|,\quad \forall\tau\in[-\tau_0,\tau_0], \tag{II.39}$$

for constants $C,\tau_0>0$. Using (II.38), we have to estimate $\|u\|_{L^2(\mathbb{R}_+)}$ and $\|H[\gamma,\xi]u\|_{L^2(\mathbb{R}_+)}$. Let $f=(H[\gamma]-\Theta(\gamma))\,u$. Then :

$$f=(\Theta(\gamma)-\Theta(\gamma+\tau))\,u+w,$$

where

$$w(t)=e^{-\tau t}\left(-2\tau\partial_t+2(\xi(\gamma+\tau)-\xi(\gamma))(t-\xi(\gamma))-(\xi(\gamma+\tau)-\xi(\gamma))^2+\tau^2\right)\varphi_\gamma(t).$$

Therefore, thanks to (II.30) and (II.34), we have :

$$\|f\|_{L^2(\mathbb{R}_+)}\leq C|\tau|,\quad \forall\tau\in[-\tau_0,\tau_0]. \tag{II.40}$$

Noticing that, after an integration by parts, $\langle f,\varphi_\gamma\rangle_{L^2(\mathbb{R}_+)}=0$, we write,

$$u=(H[\gamma]-\Theta(\gamma))^{-1}f.$$

It is a standard result that the operator norm of $(H[\gamma]-\Theta(\gamma))^{-1}$ is bounded on the orthogonal space of φ_γ and is estimated by the inverse of the gap between the first two eigenvalues of $H[\gamma]$. Therefore, thanks to (II.40), we get that $\|u\|_{L^2(\mathbb{R}_+)}\leq\widetilde{C}|\tau|$ for some constant $\widetilde{C}>0$. Plugging this estimate together with (II.40) in (II.38), we get (II.39).□

In the next proposition we have a regularity result with respect to the two variables (γ,ξ).

Proposition II.7 *The functions* $(\gamma,\xi)\mapsto\mu^{(1)}(\gamma,\xi)$ *and* $(\gamma,\xi)\mapsto\varphi_{\gamma,\xi}$ *are of class* C^∞ *in* $\mathbb{R}^2$*. Moreover, we have :*

$$\partial_\gamma\mu^{(1)}(\gamma,\xi)=|\varphi_{\gamma,\xi}(0)|^2. \tag{II.41}$$

Using Proposition II.5 and Remark II.6, Proposition II.7 is sufficient for achieving the proof of Proposition II.4.

Proof of Proposition II.7. In order to reduce the problem to a problem of an operator with a fixed domain, we define the bounded operator $V[\gamma]$ on $L^2(\mathbb{R}_+)$ by :

$$V[\gamma]u=e^{\gamma\chi(t)t}u,\quad \forall u\in L^2(\mathbb{R}_+).$$

We define then the operator $\tilde{H}[\gamma,\xi]$ by :

$$D(\tilde{H}[\gamma,\xi])=\{u\in B^2(\mathbb{R}_+);u'(0)=0\},$$
$$\tilde{H}[\gamma,\xi]=V[-\gamma]H[\gamma,\xi]V[\gamma].$$

Note that the domain of $\tilde{H}[\gamma,\xi]$ is independent of γ and ξ. Note also that $\tilde{H}[\gamma,\xi]$ is not self-adjoint but it has the same spectrum as $H[\gamma,\xi]$. A fundamental state of $\tilde{H}[\gamma,\xi]$ is given by :

$$\tilde{\varphi}_{\gamma,\xi}(t)=V[-\gamma]\varphi_{\gamma,\xi}(t).$$

We denote by $\varphi^*_{\gamma,\xi}$ the orthogonal projector on $\varphi_{\gamma,\xi}$. Let us consider a point (γ_0,ξ_0). We define the operator $M_0:D(\tilde{H}[\gamma,\xi])\times\mathbb{C}\longrightarrow L^2(\mathbb{R}_+)\times\mathbb{C}$ by :

$$M_0:=\begin{pmatrix}\tilde{H}[\gamma_0,\xi_0]-\mu_0 & \tilde{\varphi}_0\\ \varphi_0^* & 0\end{pmatrix},$$

where $\mu_0=\mu^{(1)}(\gamma_0,\xi_0)$ and $\varphi_0=\varphi_{\gamma_0,\xi_0}$.

The operator M_0 is invertible and its inverse R_0 is given by :

$$R_0=\begin{pmatrix}E_0 & E_0^+\\ E_0^- & E_0^{+-}\end{pmatrix},$$

where the coefficients of R_0 are :

$$E_0 = V[-\gamma_0]\tilde{R}[\gamma_0, \xi_0]V[\gamma_0], \tag{II.42}$$

$$E_0^+ = V[-\gamma_0]\varphi_0, \tag{II.43}$$

$$E_0^- = \varphi_0^* V[\gamma_0], \tag{II.44}$$

$$E_0^{+-} = 0. \tag{II.45}$$

The operator $\tilde{R}[\gamma_0, \xi_0]$ is the regularized resolvent which is equal to 0 on $\mathbb{R} \cdot \varphi_0$ and to $(H[\gamma_0, \xi_0] - \mu_0)^{-1}$ on $\varphi_0^\perp$.

Now we define, in a neighborhood of (γ_0, ξ_0, μ_0) the operator $M(\gamma, \xi, \mu)$ by :

$$M(\gamma, \xi, \mu) = \begin{pmatrix} \tilde{H}[\gamma, \xi] - \mu & \tilde{\varphi}_0 \\ \varphi_0^* & 0 \end{pmatrix}.$$

The operator $M(\gamma, \xi, \mu)$ is also invertible in a neighborhood of (γ_0, ξ_0, μ_0) and we denote its inverse by :

$$R(\gamma, \xi, \mu) = \begin{pmatrix} E(\gamma, \xi, \mu) & E^+(\gamma, \xi, \mu) \\ E^-(\gamma, \xi, \mu) & E^{+-}(\gamma, \xi, \mu) \end{pmatrix}.$$

It is then standard to prove the following two points (cf. Ref. 18 for details) :

- The coefficients of $R(\gamma, \xi, \mu)$ are C^∞ in a neighborhood of (γ_0, ξ_0, μ_0).
- A number μ is an eigenvalue of $H[\gamma, \xi]$ if and only if $E^{+-}(\gamma, \xi, \mu) = 0$.

Moreover, in a neighborhood of (γ_0, ξ_0, μ_0), if μ is an eigenvalue of $H[\gamma, \xi]$, then $V[\gamma]E^+(\gamma, \xi, \mu)$ is a corresponding eigenfunction.

Thus, in a neighborhood of (γ_0, ξ_0), the eigenvalues of the operator $H[\gamma, \xi]$ are given by the solutions of the equation $E^{+-}(\gamma, \xi, \mu) = 0$. By viewing the operator $M(\gamma, \xi, \mu)$ as a perturbation of M_0, we can calculate the coefficients of $R(\gamma, \xi, \mu)$ and we obtain that :

$$\partial_\mu E^{+-}(\gamma_0, \xi_0, \mu_0) = 1.$$

As the function $E^{+-}(\gamma, \xi, \mu)$ is of class C^∞, we can apply the implicit function theorem and get the existence of a number $\eta > 0$ and a function μ of class C^∞ such that :

$$\forall (\gamma, \xi) \in]\gamma_0 - \eta, \gamma_0 + \eta[\times]\xi_0 - \eta, \xi_0 + \eta[, \quad \forall \mu \in]\mu_0 - \eta, \mu_0 + \eta[,$$

$$E^{+-}(\gamma, \xi, \mu) = 0 \Leftrightarrow \mu = \mu(\gamma, \xi).$$

This proves that the functions $(\gamma,\xi)\mapsto\mu^{(1)}(\gamma,\xi)$ and $(\gamma,\xi)\mapsto\varphi_{\gamma,\xi}$ are of class C^∞. $\square$

C. Asymptotic behavior

The asymptotic behavior at $\pm\infty$ of the eigenvalue $\Theta(\gamma)$ with respect to the parameter γ is given in the following proposition.

Proposition II.8 *There exist constants $C_0,\gamma_0>0$ such that the eigenvalue $\Theta(\gamma)$ satisfies :*

$$1-C_0\gamma\exp(-\gamma^2)\leq\Theta(\gamma)<1,\quad\forall\gamma\in[\gamma_0,+\infty[,\tag{II.46}$$

and

$$-\gamma^2\leq\Theta(\gamma)\leq-\gamma^2+\frac{1}{4\gamma^2},\quad\forall\gamma\in]-\infty,0[.\tag{II.47}$$

Proof. We prove the estimate (II.46). Note that by the min-max principle and Theorem II.2 we get for any $\gamma>0$:

$$\mu^{(1)}(0,\xi(\gamma))\leq\Theta(\gamma)<1.\tag{II.48}$$

The following estimate for the Neumann problem is obtained by Bolley-Helffer[17] (formula (A.18)) :

$$\left|\mu^{(1)}(0,\xi)-1\right|\leq C\xi\exp-\xi^2,\quad\forall\xi\in[A,+\infty[,$$

where $C,A>0$ are constants independent of ξ. Recalling (II.15), the last estimate gives :

$$\left|\mu^{(1)}(0,\xi(\gamma))-1\right|\leq C_0\gamma\exp-\gamma^2,\quad\forall\gamma\in[\gamma_0,+\infty[,$$

where $\gamma_0=\max(\sqrt{A},1)$ and $C_0=2C$. Upon substitution in (II.48), we arrive at the estimate (II.46).

The relation (II.15) gives the lower bound $\Theta(\gamma)\geq-\gamma^2$. To get the upper bound in (II.47), we use the function $e^{\gamma t}$ (with $\gamma<0$) as a trial function for the quadratic form defining $H[\gamma,0]$, this which gives,

$$\frac{q[\gamma,0](e^{\gamma t})}{\|e^{\gamma t}\|^2_{L^2(\mathbb{R}_+)}}\leq-\gamma^2+\frac{1}{4\gamma^2},\quad\forall\gamma\in]-\infty,0[.$$

Therefore, we get by the min-max principle that $\mu^{(1)}(\gamma,0) \leq -\gamma^2 + \dfrac{1}{4\gamma^2}$. Recalling (II.6), we get the upper bound in (II.47). $\square$

D. Exponential decay of the ground state

Using Agmon's technique (cf. Ref. 15), we get the following decay result for the eigenfunction φ_γ.

Proposition II.9 *For each $\epsilon \in]0,1[$ there is a positive constant C_ϵ such that, for all $\gamma \in \mathbb{R}$, we have the following estimate for the eigenfunction φ_γ :*

$$\left\| \exp\left(\epsilon \frac{(t-\xi(\gamma))^2}{2}\right) \varphi_\gamma \right\|_{H^1(\{t\in\mathbb{R}_+;(t-\xi(\gamma))\geq C_\epsilon\})} \leq C_\epsilon(1+\gamma_- + \gamma_-^2), \tag{II.49}$$

where we use the notation $\gamma_- = \max(-\gamma, 0)$.

Proof. Let us consider a function $\Phi \in H^1(\mathbb{R}_+)$. Given an integer $N \in \mathbb{N}$, an integration by parts gives the following identity :

$$\begin{aligned}
&\int_0^N \left[\left| \left(e^\Phi \varphi_\gamma\right)' \right|^2 + \left| (t-\xi(\gamma)) e^\Phi \varphi_\gamma \right|^2 \right] dt + \gamma \left| e^{\Phi(0)} \varphi_\gamma(0) \right|^2 - \varphi_\gamma'(N) e^{2\Phi(N)} \varphi_\gamma(N) \\
&= \Theta(\gamma) \left\| e^\Phi \varphi_\gamma \right\|^2_{L^2([0,N])} + \left\| \Phi' e^\Phi \varphi_\gamma \right\|^2_{L^2([0,N])} .
\end{aligned} \tag{II.50}$$

Let us recall that the eigenfunction φ_γ is strictly positive. It results then from the eigenvalue equation satisfied by φ_γ :

$$\varphi_\gamma''(t) = \left((t-\xi(\gamma))^2 - \Theta(\gamma)\right) \varphi_\gamma(t) > 0, \quad \forall t \in]\sqrt{\Theta(\gamma)} + \xi(\gamma), +\infty[.$$

Therefore, the function φ_γ' is increasing on $]\sqrt{\Theta(\gamma)} + \xi(\gamma), +\infty[$. On the other hand, as $\varphi_\gamma \in H^2(\mathbb{R}_+)$, the Sobolev imbedding theorem gives $\lim\limits_{t\to+\infty} \varphi_\gamma'(t) = 0$. Thus, combining with the monotonicity of φ_γ', we get finally that :

$$\varphi_\gamma'(t) < 0, \quad \forall t \in]\sqrt{\Theta(\gamma)} + \xi(\gamma), +\infty[.$$

Taking $N > \sqrt{\Theta(\gamma)} + \xi(\gamma)$ and recalling that $\Theta(\gamma) < 1$, the identity (II.50) yields the estimate :

$$\begin{aligned}&\int_0^N \left[\left|\left(e^{\Phi}\varphi_\gamma\right)'\right|^2 + \left|(t-\xi(\gamma))e^{\Phi}\varphi_\gamma\right|^2\right] dt + \gamma\left|e^{\Phi(0)}\varphi_\gamma(0)\right|^2 \\ &\leq \left\|e^{\Phi}\varphi_\gamma\right\|^2_{L^2([0,N])} + \left\|\Phi' e^{\Phi}\varphi_\gamma\right\|^2_{L^2([0,N])}. \end{aligned} \tag{II.51}$$

To estimate the boundary term in (II.51), we recall that (II.38) (with $u = \varphi_\gamma$ and $\xi = \xi(\gamma)$) gives :

$$|\varphi_\gamma(0)|^2 \leq 2 + (\gamma_-)^2.$$

Therefore, the estimate (II.51) becomes :

$$\int_0^N \left[\left|\left(e^{\Phi}\varphi_\gamma\right)'\right|^2 + \left((t-\xi(\gamma))^2 - |\Phi'|^2 - 1\right)\left|e^{\Phi}\varphi_\gamma\right|^2\right] dt \leq \gamma_-\sqrt{2+(\gamma_-)^2}e^{2\Phi(0)}. \tag{II.52}$$

Now we take Φ as :

$$\Phi(t) = \epsilon\frac{(t-\xi(\gamma))^2}{2}.$$

We can then rewrite (II.52) as :

$$\int_{t\in[0,N],(t-\xi(\gamma))\geq a_\epsilon} \left[\left|\left(e^{\Phi}\varphi_\gamma\right)'\right|^2 + \left|e^{\Phi}\varphi_\gamma\right|^2\right] dt \leq \gamma_-\sqrt{2+(\gamma_-)^2}e^{\epsilon\xi(\gamma)} + e^{\epsilon a_\epsilon}, \tag{II.53}$$

where $a_\epsilon > 0$ satisfies :

$$a_\epsilon^2 - \epsilon^2 a_\epsilon - 1 \geq 1.$$

Notice that the first term on the right hand side of (II.53) is effective only if $\gamma < 0$. Coming back to the regularity of the function $\Theta(\gamma)$, the decay of $\Theta(\gamma)$ in (II.47) and the relation (II.15), we get that the function $\xi(\gamma)$ is bounded for $\gamma < 0$. Let us now take :

$$C_0 = \sup_{\gamma<0}\xi(\gamma), \quad C_\epsilon = \max\left(a_\epsilon, e^{\epsilon C_0}, e^{\epsilon a_\epsilon}\right).$$

The estimate (II.53) reads now as :

$$\int_{t\in[0,N],(t-\xi(\gamma))\geq C_\epsilon} \left[\left|\left(e^{\Phi}\varphi_\gamma\right)'\right|^2 + \left|e^{\Phi}\varphi_\gamma\right|^2\right] dt \leq C_\epsilon(1+\gamma_- + (\gamma_-)^2).$$

Noticing that the above estimate is uniform with respect to N, we get (II.49) upon passing to the limit $N \to +\infty$. $\square$

Let us now recall that the regularized resolvent $\tilde{R}[\gamma]$ is the bounded operator defined on $L^2(\mathbb{R}_+)$ by :

$$\tilde{R}[\gamma]\phi = \begin{cases} 0 & ; \ \phi \parallel \varphi_\gamma, \\ (H[\gamma] - \Theta(\gamma))^{-1}\phi & ; \ \phi \perp \varphi_\gamma, \end{cases} \tag{II.54}$$

and extended by linearity. Again, using the Agmon's technique, we get that this regularized resolvent is uniformly continuous in suitable weighted spaces.

Proposition II.10 *For each $\delta \in]0,1[$ and $\eta_0 > 0$, there exist positive constants C_0, t_0 such that,*

$$\forall \gamma \in [-\eta_0, \eta_0], \quad \forall u \in L^2\left(\mathbb{R}_+; e^{\delta(t-\xi(\gamma))}dt\right), \quad u \perp \varphi_\gamma,$$

we have,

$$\left\| e^{\delta(t-\xi(\gamma))}\tilde{R}[\gamma]u \right\|_{H^1([t_0,+\infty[)} \leq C_0 \left\| e^{\delta(t-\xi(\gamma))}u \right\|_{L^2(\mathbb{R}_+)}. \tag{II.55}$$

III. PROOF OF THEOREM I.1

In this section we prove Theorem I.1 by comparing with the basic model introduced in the preceding section. We introduce a coordinate system (s,t) near the boundary $\partial\Omega$ where t measures the distance to $\partial\Omega$ and s measures the distance in $\partial\Omega$ (cf. Appendix A).

Proposition III.1 *(Upper bound)*
Under the hypothesis of Theorem I.1, there exist positive constants C and h_0 such that, $\forall h \in]0, h_0]$, we have :

$$\mu^{(1)}(\alpha,\gamma,h) \leq h\Theta\left(h^{\alpha-1/2}(\gamma_0 + Ch^{1/2})\right) + Ch^{3/2}. \tag{III.1}$$

Proof. We start with the easy case when $\alpha < \frac{1}{2}$ and $\gamma_0 > 0$. Notice that in this case, given a constant $C > 0$, formula (II.46) gives the existence of $h_0 > 0$ such that :

$$\left|\Theta\left(h^{\alpha-1/2}(\gamma_0 + Ch^{1/2})\right) - 1\right| \leq \exp(-h^{2\alpha-1}), \quad \forall h \in]0, h_0]. \tag{III.2}$$

By comparing with the Dirichlet realization, the min-max principle gives :

$$\mu^{(1)}(\alpha,\gamma,h) \leq \lambda^{(1)}(h),$$

where $\lambda^{(1)}(h)$ is the first eigenvalue of the Dirichlet realization (on Ω) of $-(h\nabla - iA)^2$. Using the following upper bound for $\lambda^{(1)}(h)$ (cf. Ref. 9) :

$$\lambda^{(1)}(h) \leq h + Ch^{3/2}, \quad \forall h \in]0,1],$$

together with (III.2), we get (III.1).

We suppose now that $\gamma_0 \leq 0$ if $\alpha < \frac{1}{2}$. Consider a point $x_0 \in \partial\Omega$ such that $\gamma(x_0) = \gamma_0$. We suppose that $x_0 = 0$ in the coordinate system (s,t) near the boundary (cf. Appendix A). Using this coordinate system we construct a trial function $u_{h,\alpha}$ supported in the rectangle $K_h =]-h^{1/4}, h^{1/4}[\times[0,t_0[$ following the idea of Helffer-Morame[10] and Bernoff-Sternberg[7]. Since x_0 is a minimum of γ, Taylor's formula up to the first order gives the existence of positive constants C_1, h_0 such that,

$$\forall h \in]0,h_0], \quad |\gamma(s) - \gamma_0| \leq C_1 h^{1/2} \quad \text{ in }]-h^{1/2}, h^{1/2}[.$$

Thus, given a trial function u supported in K_h, we have the following estimate :

$$q_{h,A,\Omega}^{\alpha,\gamma}(u) \leq q_{h,A,\Omega}^{\alpha,\tilde{\gamma}_0}(u), \quad \forall h \in]0,h_0], \tag{III.3}$$

where $\tilde{\gamma}_0 = \gamma_0 + C_1 h^{1/2}$. So it is enough to work with $q_{h,A,\Omega}^{\alpha,\tilde{\gamma}_0}$.

We introduce $\eta = h^{\alpha-1/2}\tilde{\gamma}_0$ and we choose now the following trial function :

$$u_{h,\alpha} = a^{-1/2} \exp\left(-i\frac{\xi(\eta)s}{h^{1/2}}\right) v_{h,\alpha}, \tag{III.4}$$

where $a(s,t) = 1 - t\kappa_{\mathrm{r}}(s)$ and

$$v_{h,\alpha} = h^{-3/8}\varphi_\eta\left(h^{-1/2}t\right)\chi(t) \times f\left(h^{-1/4}s\right). \tag{III.5}$$

The function χ is a cut-off equal to 1 in a compact interval $[0, t_0/2]$ and the function $f \in C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[;\mathbb{R})$ is chosen such that $\|f\|_{L^2(\mathbb{R})} = 1$.

Note that the decay of φ_η in Proposition II.9 gives[29] : For every $\delta > 0$ and $k \in \mathbb{N}$, there exist positive constants $C_{k,\delta}$ and h_0 such that,

$$\int_{\mathbb{R}_+} t^k |\varphi_\eta(t)|^2 dt \leq C_{k,\delta} h^{-\delta k}, \quad \forall h \in]0,h_0]. \tag{III.6}$$

We work with the choice of gauge given in Proposition A.2. Using formula (A.3), we can write :

$$q_{h,A,\Omega}^{\alpha,\tilde{\gamma}_0}(u_{h,\alpha}) = \int_{-|\partial\Omega|/2}^{|\partial\Omega|/2} \int_0^{t_0} \left[|h\partial_t v_{h,\alpha}|^2 + a^{-2} \left| \left(h^{1/2}\xi(\eta) - t\left(1 - \frac{t}{2}\kappa_{\mathrm{r}}(s)\right)\right) v_{h,\alpha} \right|^2 \right] dsdt$$
$$+ h^{3/2}\eta \int_{-|\partial\Omega|/2}^{|\partial\Omega|/2} |v_{h,\alpha}(s,0)|^2 ds$$
$$+ h^2 \int_{-|\partial\Omega|/2}^{|\partial\Omega|/2} \int_0^{t_0} \left[|(\partial_t a^{-1/2}) v_{h,\alpha}|^2 + 2a^{-1/2}(\partial_t a^{-1/2})(\partial_t v_{h,\alpha}) v_{h,\alpha} + a^{-2}|\partial_s v_{h,\alpha}|^2 \right] a\, dsdt. \tag{III.7}$$

Recalling the expression of $v_{h,\alpha}$ (cf. (III.5)), we can replace the function χ by 1 getting an exponentially small error on the right hand side of (III.7), thanks to the decay of φ_η in Proposition II.9. After a change of variables and using the decay of φ_η in (III.6), the leading order term on the right hand side of (III.7) is equal to :

$$h\left(\int_0^{+\infty} \left[|\varphi_\eta'(t)|^2 + |(t - \xi(\eta))\varphi_\eta|^2 dt \right] dt + \eta|\varphi_\eta(0)|^2 \right),$$

and the error is of order $\mathcal{O}(h^{3/2})$. Therefore, we get constants $C, h_0 > 0$ such that :

$$\left| q_{h,A,\Omega}^{\alpha,\tilde{\gamma}_0}(u_{h,\alpha}) - h\Theta(\eta) \right| \leq Ch^{3/2}, \quad \forall h \in]0, h_0].$$

Using formula (A.4) and the decay of φ_η (Proposition II.9), we obtain that the L^2 norm of $u_{h,\alpha}$ is exponentially close to 1 as $h \to 0$. The application of the min-max principle permits now to prove (III.1). $\square$

Remark III.2 *In the regime $\alpha \in]\frac{1}{2}, 1[$, we have, thanks to Proposition II.5 :*

$$\Theta\left(h^{\alpha-1/2}\gamma_0\right) = \Theta_0 + 6M_3 h^{\alpha-1/2} + \mathcal{O}(h^{2\alpha-1}).$$

Substituting the above expansion in the upper bound (III.1), we get the following upper bound for the eigenvalue $\mu^{(1)}(\alpha, \gamma, h)$,

$$\mu^{(1)}(\alpha, \gamma, h) \leq h\Theta_0 + 6M_3\gamma_0 h^{\alpha+1/2} + \mathcal{O}(h^{\inf(3/2, 2\alpha)}).$$

We shall prove that this upper bound is actually an asymptotic expansion of $\mu^{(1)}(\alpha, \gamma, h)$ as h tends 0 (see Remark V.11).

Proposition III.3 *(Lower bound)*

Under the hypothesis of Theorem I.1, there exist positive constants C, C' and h_0 such that, $\forall h \in]0, h_0]$, we have :

$$\mu^{(1)}(\alpha,\gamma,h) \geq h\Theta\left(h^{\alpha-1/2}\gamma_0(1+C'h^{1/4})\right) - Ch^{5/4}. \tag{III.8}$$

Proof. We follow the technique of Ref. 10 and we localize by means of a partition of unity to compare with the model operators in $\mathbb{R}^2$ and $\mathbb{R}\times\mathbb{R}_+$. Let us explain the heuristic idea. A partition of unity permits to estimate the quadratic form $q^{\alpha,\gamma}_{h,A,\Omega}$ locally in small subsets of Ω. Near the boundary, we obtain after a transformation of coordinates that the expression of $q^{\alpha,\gamma}_{h,A,\Omega}$ is to leading order asymptotics as that of the half-plane model. In the interior of Ω, the expression of the quadratic form is actually like that of the entire-plane model.

Let us introduce a partition of unity (χ_j) of $\mathbb{R}^2$ that satisfies

$$\sum_j |\chi_j|^2 = 1, \quad \sum_j |\nabla\chi_j|^2 < +\infty, \quad \mathrm{supp}\chi_j \subset D(z_j,1),$$

where for $z\in\mathbb{R}^2$ and $r>0$, we denote by $D(z,r)$ the disk of center z and radius r.

We introduce now the scaled partition of unity :

$$\chi_j^h(z) := \chi_j(\epsilon_0 h^\rho z), \quad \forall z \in \mathbb{R}^2,$$

where ϵ_0 and ρ are two positive numbers to be chosen suitably. Note that (χ_j^h) now satisfies :

$$\sum_j |\chi_j^h|^2 = 1, \tag{III.9}$$

$$\sum_j |\nabla\chi_j^h|^2 \leq C\epsilon_0^{-2}h^{-2\rho}, \tag{III.10}$$

$$\mathrm{supp}\ \chi_j^h \subset Q_j^h := D(z_j^h, \epsilon_0 h^\rho), \tag{III.11}$$

where C is a positive constant. We can also suppose that :

$$\text{either supp } \chi_j^h \cap \partial\Omega = \emptyset \quad \text{or } z_j^h \in \partial\Omega. \tag{III.12}$$

Note that the alternative in (III.12) permits us to write the sum in (III.9) under the form :

$$\sum = \sum_{int} + \sum_{bnd},$$

where the summation over "*int*" means that the support of χ_j^h do not meet the boundary while that over "*bnd*" means the converse.

We have now the following decomposition formula :

$$q_{h,A}^{\alpha,\gamma}(u) = \sum_j q_{h,A}^{\alpha,\gamma}(\chi_j^h u) - h^2 \sum_j \| \, |\nabla \chi_j^h| \, u\|^2, \quad \forall u \in H^1(\Omega), \tag{III.13}$$

usually called the IMS formula (cf. Ref. 24). We have now to bound from below each of the terms on the right hand side of (III.13). Note that (III.10) permits to estimate the contribution of the last term in (III.13) :

$$h^2 \sum_j \| \, |\nabla \chi_j^h| \, u\|^2 \leq C\epsilon_0^{-2} h^{2-2\rho} \|u\|^2, \quad \forall u \in H^1(\Omega). \tag{III.14}$$

If χ_j^h is supported in Ω, then we have :

$$q_{h,A}^{\alpha,\gamma}(\chi_j^h u) = \int_{\mathbb{R}^2} |(h\nabla - iA)\chi_j^h u|^2 dx.$$

Since the lowest eigenvalue of the Schrödinger operator with constant magnetic field in $\mathbb{R}^2$ is equal to h, we get :

$$q_{h,A}^{\alpha,\gamma}(\chi_j^h u) \geq h \int_\Omega |\chi_j^h u|^2 dx, \quad \forall u \in H^1(\Omega). \tag{III.15}$$

We have now to estimate $q_{h,A,\Omega}^{\alpha,\gamma}(\chi_j^h u)$ when χ_j^h meets the boundary. It is in this case that we see the effect of the boundary condition. Note that, by writing $q_{h,A,\Omega}^{\alpha,\gamma}(\chi_j^h u)$ in the boundary coordinates, thanks to Proposition A.1, there exists a positive constant C_1 independent of h and j such that :

$$\int_\Omega |(h\nabla - iA)\chi_j^h u|^2 dx \geq (1 - C_1\epsilon_0 h^\rho) \int_{\mathbb{R}\times\mathbb{R}_+} |(h\nabla - i\tilde{A})\chi_j^h u|^2 ds dt, \quad \forall u \in H^1(\Omega), \tag{III.16}$$

where $\tilde{A}$ is the vector field associated to A by (A.2).

By a gauge transformation, we get a new magnetic potential $\tilde{A}_{new,j}$ satisfying :

$$\tilde{A}_{new,j} = \tilde{A} - \nabla \phi_j^h,$$

$$\tilde{A}_{new,j}(z_j^h) = 0,$$

$$\left|\tilde{A}_{new,j}(w) - \tilde{A}_{lin}^j(w)\right| \leq C|w|^2, \quad w = (s,t), \tag{III.17}$$

where $A_{lin}^j := \frac{1}{2}(-t, s)$ is the linear magnetic potential and $C > 0$ is a constant independent of h and j.

Given $\theta > 0$ and any function v of support in $\mathbb{R} \times \mathbb{R}_+$, we get by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left|\int_{\mathbb{R}\times\mathbb{R}_+} (h\nabla - i\tilde{A}_{new,j})v \cdot \overline{(\tilde{A}_{new,j} - \tilde{A}_{lin}^j)v}\, dsdt\right| \\
&\leq h^{2\theta} \int_{\mathbb{R}\times\mathbb{R}_+} |(h\nabla - i\tilde{A}_{new,j})v|^2 + h^{-2\theta} \int_{\mathbb{R}\times\mathbb{R}_+} |(\tilde{A}_{new,j} - \tilde{A}_{lin}^j)v|^2.
\end{aligned}$$

Writing $\tilde{A}_{new,j} = \tilde{A}_{lin}^j + (\tilde{A}_{new,j} - \tilde{A}_{lin}^j)$ and using (III.17), we get a positive constant $\tilde{C}$ independent of h and j such that :

$$\int_{\mathbb{R}\times\mathbb{R}_+} |(h\nabla - i\tilde{A}_{new,j})v|^2 dsdt \geq (1 - h^{2\theta}) \int_{\mathbb{R}\times\mathbb{R}_+} \left|(h\nabla - i\tilde{A}_{lin}^j)v\right|^2 dsdt - \tilde{C}h^{-2\theta} \left\| |w|^2 \chi_j^h u\right\|^2. \tag{III.18}$$

Let us recall that $\chi_j^h u$ is supported in the disk $D(z_j^h, \epsilon_0 h^\rho)$. Upon noticing that

$$\int_{\mathbb{R}\times\mathbb{R}_+} |(h\nabla - i\tilde{A})\chi_j^h u|^2 dsdt = \int_{\mathbb{R}\times\mathbb{R}_+} \left|(h\nabla - i\tilde{A}_{new,j}) \exp\left(-i\frac{\phi_j^h}{h}\right) \chi_j^h u\right|^2 dsdt,$$

we get by combining (III.18) (with $v = \exp\left(-i\frac{\phi_j^h}{h}\right) \chi_j^h u$) together with (III.16), a constant $C_2 > 0$ such that :

$$\begin{aligned}
\int_{\mathbb{R}\times\mathbb{R}_+} |(h\nabla - iA)\chi_j^h u|^2 dx \;\geq\; &(1 - C_2\epsilon_0 h^\rho - C_2 h^{2\theta}) \int_{\mathbb{R}\times\mathbb{R}_+} \left|(h\nabla - i\tilde{A}_{lin}^j) \exp\left(-i\frac{\phi_j^h}{h}\right) \chi_j^h u\right|^2 dsdt \\
&- C_2\epsilon_0 h^{4\rho - 2\theta} \int_{\mathbb{R}\times\mathbb{R}_+} |\chi_j^h u|^2 dsdt.
\end{aligned} \tag{III.19}$$

Notice also, (possibly changing C_2), we have in $D(z_j^h, \epsilon_0 h^\rho)$,

$$\gamma(x) \geq \gamma(z_j^h) - C_2\epsilon_0 h^\rho.$$

Then, by putting,

$$\tilde{\gamma}_j = \frac{\gamma(z_j^h) - C_2\epsilon_0 h^\rho}{1 - C_2 h^{2\theta} - C_2\epsilon_0 h^\rho},$$

the estimate (III.19) reads finally :

$$q^{\alpha,\gamma}_{h,A,\Omega}(\chi_j^h u) \;\geq\; (1 - C_2\epsilon_0 h^{\rho} - C_2 h^{2\theta}) q^{\alpha,\tilde{\gamma}_j}_{h,\tilde{A}^j_{lin},\mathbb{R}\times\mathbb{R}_+}\left(\exp\left(-i\frac{\phi_j^h}{h}\right)\chi_j^h u\right)$$
$$-C_2\epsilon_0^2 h^{4\rho-2\theta}\|\chi_j^h u\|^2. \tag{III.20}$$

Note that this permits to compare with the half-plane model operator and to get finally the energy estimate (cf. (I.10)) :

$$q^{\alpha,\gamma}_{h,A,\Omega}(\chi_j^h u) \geq \left\{(1 - C_2\epsilon_0 h^{\rho} - C_2 h^{2\theta}) h\Theta(h^{\alpha-1/2}\tilde{\gamma}_j) - C_2\epsilon_0^2 h^{4\rho-2\theta}\right\}\|\chi_j^h u\|^2_{L^2(\Omega)}. \tag{III.21}$$

We substitute now the estimates (III.8), (III.14), (III.15) in (III.13) and get finally :

$$q^{\alpha,\gamma}_{h,A,\Omega}(u) \geq h\sum_{int}\int_\Omega |\chi_j^h u|^2 dx + h\sum_{bnd}\Theta\left(h^{\alpha-1/2}\tilde{\gamma}_j\right)\int_\Omega |\chi_j^h u|^2 dx$$
$$-C\left(h^{4\rho-2\theta} + \epsilon_0^{-2}h^{2-2\rho} + h^{1+\rho} + h^{1+2\theta}\right)\|u\|^2, \quad \forall u \in H^1(\Omega). \tag{III.22}$$

As γ_0 is the minimum of γ, we can replace (III.20) by the estimate :

$$q^{\alpha,\gamma}_{h,A,\Omega}(\chi_j^h u) \;\geq\; (1 - C_2\epsilon_0 h^{\rho} - C_2 h^{2\theta}) q^{\alpha,\tilde{\gamma}_0}_{h,\tilde{A}^j_{lin},\mathbb{R}\times\mathbb{R}_+}\left(\exp\left(-i\frac{\phi_j^h}{h}\right)\chi_j^h u\right)$$
$$-C_2\epsilon_0^2 h^{4\rho-2\theta}\|\chi_j^h u\|^2, \tag{III.23}$$

where $\tilde{\gamma}_0$ is defined by :

$$\tilde{\gamma}_0 := \frac{\gamma_0}{1 - C_2 h^{2\theta} - C_2\epsilon_0 h^{\rho}}.$$

We then get instead of (III.22) :

$$q^{\alpha,\gamma}_{h,A,\Omega}(u) \geq h\sum_{int}\int_\Omega |\chi_j^h u|^2 dx + h\Theta\left(h^{\alpha-1/2}\tilde{\gamma}_0\right)\sum_{bnd}\int_\Omega |\chi_j^h u|^2 dx$$
$$-C\left(h^{4\rho-2\theta} + \epsilon_0^{-2}h^{2-2\rho} + h^{1+\rho} + h^{1+2\theta}\right)\|u\|^2, \quad \forall u \in H^1(\Omega). \tag{III.24}$$

The advantage of (III.22) is that it gives a lower bound of the quadratic form $q^{\alpha,\gamma}_{h,A,\Omega}$ in terms of a potential, see however Section IV.

We choose now $\epsilon_0 = 1$, and we optimize by taking $2 - 2\rho = 1 + \rho = 4\rho - 2\theta$ (i.e. $\rho = 3/8$ and $\theta = 1/8$) in (III.24). We obtain then (III.8) by applying the min-max principle. □

Proof of Theorem I.1. The proof follows in principle from Propositions III.1 and III.3. Actually, in the regime $\alpha < \frac{1}{2}$, we use further Proposition II.8, while in the regime $\alpha \geq \frac{1}{2}$, we use the continuity of the function $\Theta(\gamma)$ (Proposition II.4). $\square$

IV. LOCALIZATION OF THE GROUND STATE

We work in this section under the hypotheses of Theorem I.3. Due to Theorem I.1 we have in this case that

$$\lim_{h\to 0} \frac{\mu^{(1)}(\alpha,\gamma,h)}{h} < 1. \tag{IV.1}$$

Then this gives, by following the same lines of the proof of Theorem 6.3 in Ref. 10, the following proposition.

Theorem IV.1 *Under the hypotheses of Theorem I.3, there exist positive constants* δ, C, h_0 *such that, for all* $h \in]0,h_0]$*, a ground state* $u_{\alpha,\gamma,h}$ *of the operator* $P^{\alpha,\gamma}_{h,A,\Omega}$ *satisfies :*

$$\left\| \exp\left(\frac{\delta d(x,\partial\Omega)}{h^\beta}\right) u_{\alpha,\gamma,h} \right\|_{L^2(\Omega)} \leq C \|u_{\alpha,\gamma,h}\|_{L^2(\Omega)}, \tag{IV.2}$$

and

$$\left\| \exp\left(\frac{\delta d(x,\partial\Omega)}{h^\beta}\right) u_{\alpha,\gamma,h} \right\|_{H^1(\Omega)} \leq C h^{-\min(1/2,\beta)} \|u_{\alpha,\gamma,h}\|_{L^2(\Omega)}, \tag{IV.3}$$

where $\beta = 1-\alpha$ *if* $\gamma_0 < 0$ *and* $\alpha < \frac{1}{2}$*, and* $\beta = 1/2$ *otherwise.*

Proof. Integrating by parts, we get for any Lipschitz function Φ :

$$\begin{aligned} q^{\alpha,\gamma}_{h,A,\Omega}\left(\exp\left(\frac{\Phi}{h^\beta}\right) u_{\alpha,\gamma,h}\right) &= \mu^{(1)}(\alpha,\gamma,h) \left\| \exp\left(\frac{\Phi}{h^\beta}\right) u_{\alpha,\gamma,h} \right\|^2_{L^2(\Omega)} \\ &\quad + h^{2-2\beta} \left\| |\nabla\Phi| \exp\left(\frac{\Phi}{h^\beta}\right) u_{\alpha,\gamma,h} \right\|^2_{L^2(\Omega)}. \end{aligned} \tag{IV.4}$$

Let $u = \exp\left(\frac{\Phi}{h^\beta}\right) u_{\alpha,\gamma,h}$. Using the lower bound for $q^{\alpha,\gamma}_{h,A,\Omega}(u)$ in (III.24) together with the upper bound for $\mu^{(1)}(\alpha,\gamma,h)$ in (III.1), we get from (IV.4) :

$$\begin{aligned}
\sum_{int}\int_\Omega \Big(1 - \Theta(h^{\alpha-1/2}(\gamma_0 + Ch^{1/2})) - C\epsilon_0^{-2}h^{1-2\rho} - Ch^{4\rho-2\theta-1} \\
-Ch^{\min(\rho,2\theta)} - h^{1-2\beta}|\nabla\Phi|^2\Big) \times |\chi_j^h u|^2 dx \\
\leq \sum_{bnd}\int_\Omega \Big(\Theta(h^{\alpha-1/2}\tilde{\gamma}_0) - \Theta(h^{\alpha-1/2}(\gamma_0 + Ch^{1/2})) \\
+ h^{\min(4\rho-2\theta-1,1-2\rho)} + h^{1-2\beta}|\nabla\Phi|^2\Big) \times |\chi_j^h u|^2 dx.
\end{aligned}$$

We choose $\rho = \beta$ so that each χ_j^h is supported in a disk of radius $\epsilon_0 h^\beta$. We choose also $\theta > 0$ such that $4\rho - 2\theta - 1 > 0$ and we define the function Φ by :

$$\Phi(x) = \delta \max\left(\mathrm{dist}(x,\partial\Omega); \epsilon_0 h^\beta\right),$$

where δ is a positive constant to be chosen appropriately. Note that $1 - \Theta(h^{\alpha-1/2}\tilde{\gamma}_0)$ decays in the following way :

$$\begin{aligned}
&\exists\, C_0, h_0 > 0 \text{ s. t.}, \forall h \in]0,h_0],\\
&1 - \Theta(h^{\alpha-1/2}\tilde{\gamma}_0) \geq C_0 h^{2\alpha-1} \text{ if } \gamma_0 < 0 \text{ and } \alpha < \frac{1}{2},\\
&1 - \Theta(h^{\alpha-1/2}\tilde{\gamma}_0) \geq C_0 \text{ otherwise.}
\end{aligned}$$

Thus we can choose ϵ_0 and δ small enough, so that we get finally the following decay :

$$\sum_{int}\int_\Omega \left|\chi_j^h \exp\frac{\Phi}{h^\beta} u_{\alpha,\gamma,h}\right|^2 dx \leq C\int_\Omega |u_{\alpha,\gamma,h}|^2 dx.$$

This actually permits to conclude (IV.2) and, thanks to (IV.4),

$$q^{\alpha,\gamma}_{h,A,\Omega}\left(\exp\frac{\delta d(x,\partial\Omega)}{h^\beta}u_{\alpha,\gamma,h}\right) \leq Ch^{\min(2-2\beta,1)}\left\|\exp\frac{\delta d(x,\partial\Omega)}{h^\beta}u_{\alpha,\gamma,h}\right\|^2_{L^2(\Omega)}. \tag{IV.5}$$

For a function $u \in H^1(\Omega)$, let $\tilde{u}(s,t)$ be defined by means of boundary coordinates (s,t) and equal to the restriction of u in Ω_{t_0} (cf. Appendix A). Notice that :

$$|\tilde{u}(s,0)|^2 = -2\int_0^\infty \left\{\partial_t\left(\chi(t)\tilde{u}(s,t)\right)\right\}\chi(t)\tilde{u}(s,t)dt,$$

where χ is the same cut-off introduced in (III.5). Integrating the above identity with respect to the variable s then applying a Cauchy-Schwarz inequality, we get after a change of variables the following interpolation inequality :

$$\|u\|^2_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)} \times \|u\|_{H^1(\Omega)},$$

where C is a positive constant depending only on Ω.
Applying again a Cauchy-Schwarz inequality, the preceding estimate gives :

$$\|(h\nabla - iA)u\|^2_{L^2(\Omega)} \leq 2q^{\alpha,\gamma}_{h,A,\Omega}(u) + Ch\|u\|^2_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega).$$

In particular, for $u = \exp\left(\frac{\delta d(x,\partial\Omega)}{h^\beta}\right) u_{\alpha,\gamma,h}$, we get (IV.3), thanks to (IV.2) and (IV.5). $\square$

We study now the decay near the boundary. Let us consider a number $\beta > 0$ and a Lipschitz function Φ_0 defined in $\overline{\Omega}$. The function Φ_0 and the number β will be chosen later in an appropriate manner. Choosing $\rho = \frac{3}{8}$, $\theta = \frac{1}{8}$ and ϵ_0 large enough, the energy estimate (III.22) together with the upper bound (III.1) give the existence of a positive constant C such that :

$$\begin{aligned} 0 \geq h\sum_{int}\int_\Omega \left(1 - \Theta\left(h^{\alpha-1/2}\tilde{\gamma}_0\right) - Ch^{1/4} - h^{1-2\beta}|\nabla\Phi_0|^2\right)\left|\exp\left(\frac{\Phi_0}{h^\beta}\right)\chi^h_j u_{\alpha,\gamma,h}\right|^2 dx \\ +h\sum_{bnd}\int_\Omega \left[\left(\Theta\left(h^{\alpha-1/2}\tilde{\gamma}(x)\right) - \Theta\left(h^{\alpha-1/2}\tilde{\gamma}_0\right)\right) - Ch^{1/4} - h^{1-2\beta}|\nabla\Phi_0|^2\right] \\ \times\left|\exp\left(\frac{\Phi_0}{h^\beta}\right)\chi^h_j u_{\alpha,\gamma,h}\right|^2 dx, \end{aligned}$$

where $\tilde{\gamma}_0 = \gamma_0 + Ch^{1/2}$. The function γ is extended to a small boundary sheath by means of boundary coordinates in the following way :

$$\gamma(x) = \gamma(s(x)), \quad \forall x \in \Omega_{t_0}.$$

In the case $\alpha < \frac{1}{2}$ and $\gamma_0 = 0$, thanks to Proposition II.8, the difference between $\Theta\left(h^{\alpha-1/2}\tilde{\gamma}(x)\right)$ and $\Theta\left(h^{\alpha-1/2}\tilde{\gamma}_0\right)$ decays in the following way :

$$\forall\varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall x \in (\gamma - \gamma_0)^{-1}([\epsilon, +\infty[),\ \Theta\left(h^{\alpha-1/2}\tilde{\gamma}(x)\right) - \Theta\left(h^{\alpha-1/2}\tilde{\gamma}_0\right) > C_\varepsilon.$$

In the case $\alpha < \frac{1}{2}$ and $\gamma_0 < 0$, we have a stronger decay :

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall x \in (\gamma - \gamma_0)^{-1}([\epsilon, +\infty[),$$
$$\Theta\left(h^{\alpha-1/2}\tilde{\gamma}(x)\right) - \Theta\left(h^{\alpha-1/2}\tilde{\gamma}_0\right) > C_\varepsilon h^{1-2\alpha}.$$

So by taking Φ_0 in the form :

$$\Phi_0(x) = \delta\chi(\mathrm{dist}(x, \partial\Omega))\mathrm{dist}\left(x, \{x \in \partial\Omega; \gamma(x) = \gamma_0\}\right),$$

with δ an appropriate positive constant and χ is the same as in (III.5), we get for each $\epsilon > 0$ the following decay near the boundary :

$$\int_{\mathrm{dist}(x,\partial\Omega)<t_0} \left|\exp\frac{\Phi_0}{h^\beta}\, u_{\alpha,\gamma,h}\right|^2 dx \leq C_\epsilon \exp\frac{\epsilon}{h^\beta}\|u_{\alpha,\gamma,h}\|^2, \quad \forall h \in]0, h_\epsilon], \tag{IV.6}$$

with $\beta = 1-\alpha$ if $\gamma_0 < 0$ and $\alpha < \frac{1}{2}$, and $\beta = \frac{1}{2}$ otherwise. This gives finally the decay in Theorem I.3.

For the critical case $\alpha = \frac{1}{2}$ and γ_0 arbitrary, we define the function Φ_0 by :

$$\Phi_0(x) = \delta\ \chi(\mathrm{dist}(x, \partial\Omega))\ \mathrm{dist}_{\mathrm{agm}}\left(x, \{x \in \partial\Omega; \gamma(x) = \gamma_0\}\right),$$

where $\mathrm{dist}_{\mathrm{agm}}$ is the Agmon distance associated to the metric $(\Theta(\gamma(x)) - \Theta(\gamma_0))_+$. We obtain then a similar decay result to (IV.6).

In the case when $\alpha > \frac{1}{2}$, we need a finer energy estimate than (III.22), see however Remark V.11.

V. TWO-TERM ASYMPTOTICS

In this section we suppose in addition to the hypotheses of Theorem I.1 that $\alpha \geq \frac{1}{2}$. We give two-term asymptotic expansions for the ground state energy showing the influence of the scalar curvature and we finish the proofs of the remaining theorems announced in the introduction.

A. Upper bound

We construct a trial function defined by means of boundary coordinates (s,t) near a point $z_0 \in \partial\Omega$. We suppose that $z_0 = 0$ in the coordinate system (s,t) and we denote by $\kappa_0 = \kappa_{\mathrm{r}}(0)$, $a_0 = 1 - t\kappa_0$ and $\eta(z_0) = h^{\alpha-1/2}\gamma(z_0)$. We then define the trial function :

$$u_h = \exp\left(-i\frac{\xi(\eta(z_0))s}{h^{1/2}}\right) v_h(s,t), \tag{V.1}$$

with

$$v_h(s,t) = h^{-5/16} a_0^{-1/2}(t)\varphi_{\eta(z_0)}\left(h^{-1/2}t\right)\chi(t)\cdot f\left(h^{-1/8}s\right), \tag{V.2}$$

and where the functions χ and f are as in (III.5).

We continue now to work in the spirit of Ref. 10. We work with the gauge given in Proposition A.2. An explicit calculation, thanks to the decay of $\varphi_{\eta(z_0)}$ (Proposition II.9), gives the following lemma.

Lemma V.1 *With the above notations, for each $\alpha \in [\frac{1}{2}, 1]$ and $\gamma \in C^\infty(\partial\Omega;\mathbb{R})$, there exist positive constants C, h_0 such that, $\forall h \in]0, h_0]$, we have the following estimate :*

$$\left| q_{h,A,\Omega}^{\alpha,\gamma(z_0)}(u_h) - \int_{\mathbb{R}_+} H^h\left(U^h\varphi_{\eta(z_0)}\right) \times \left(U^h\varphi_{\eta(z_0)}\right) dt \right| \leq Ch^{13/8}, \tag{V.3}$$

where the operators H^h and U^h are defined respectively by :

$$H^h = a_0^{-2}\left(t\left(1 - t\frac{\kappa_0}{2}\right) - h^{1/2}\xi(\eta(z_0))\right)^2 - h^2 a_0^{-1}\partial_t(a_0\partial_t),$$

$$(U^h g)(t) = h^{-1/4}g(h^{-1/2}t), \quad \forall g \in L^2(\mathbb{R}_+).$$

Proof. Note that in the support of u_h we have[30]

$$a = a_0 + \mathcal{O}(h^{5/8}), \quad \tilde{A}_1 = -t\left(1 - \frac{t}{2}\kappa_0\right) + \mathcal{O}(h^{9/8}).$$

Then, thanks to formula (A.3) (also cf. (III.7)) and the decay of $\varphi_{\eta(z_0)}$ (Proposition II.9), we get modulo $\mathcal{O}(h^{13/8})$:

$$\begin{aligned} q_{h,A,\Omega}^{\alpha,\gamma(z_0)}(u_h) \; &= \int_{\mathbb{R}\times\mathbb{R}_+} a_0\left\{|h\partial_t v_h|^2 + a_0^{-2}\left|\left(t\left(1 - \frac{t}{2}\kappa_0\right) - h^{1/2}\xi(\eta(z_0))\right)v_h\right|^2\right\} dsdt \\ &+ h^{3/2}\eta(z_0)\int_{\mathbb{R}} |v_h(s,0)|^2. \end{aligned} \tag{V.4}$$

Integrating with respect to s, the right hand side above is equal to :

$$h^{-1/2}\int_{\mathbb{R}} a_0 \left\{ h^2 \left|\partial_t \left(a_0^{-1/2}\varphi_{\eta(z_0)}(h^{-1/2}t)\chi(t)\right)\right|^2 \right.$$
$$\left. +a_0^{-3}\left|\left(t\left(1-\frac{t}{2}\kappa_0\right) - h^{1/2}\xi(\eta(z_0))\right)\varphi_{\eta(z_0)}(h^{-1/2}t)\chi(t)\right|^2\right\} dsdt + h\eta(z_0)|\varphi_{\eta(z_0)}(0)|^2.$$

We can replace the function χ in the above expression by 1 getting an exponentially small error, thanks to Proposition II.9. Thus, modulo a small exponential error, we rewrite the above expression as :

$$\int_{\mathbb{R}}\left\{h^2 a_0\left|\partial_t\left(U^h\varphi_{\eta(z_0)}\right)\right|^2 + a_0^{-2}\left|\left(t\left(1-\frac{t}{2}\kappa_0\right)-h^{1/2}\xi(\eta(z_0))\right)\left(U^h\varphi_{\eta(z_0)}\right)\right|^2\right\} dsdt$$
$$+h^{3/2}\eta(z_0)\left|\left(U^h\varphi_{\eta(z_0)}\right)(0)\right|^2.$$

Notice that we have the boundary condition $(U^h\varphi_{\eta(z_0)})'(0) = h^{-1/2}\eta(z_0)(U^h\varphi_{\eta(z_0)})(0)$. Therefore, integrating by parts, the above expression is equal to $\int_{\mathbb{R}_+} H^h\left(U^h\varphi_{\eta(z_0)}\right)\times\left(U^h\varphi_{\eta(z_0)}\right)dt$. Upon substituting in (V.3), this finishes the proof of the lemma. $\square$

Similar computations give also the following lemma.

Lemma V.2 *Under the hypotheses of Lemma V.1, there exist positive constants C, h_0 such that, $\forall h \in]0, h_0]$, we have :*

$$\left\|(H^h - H_0^h - H_1^h)U^h\varphi_{\eta(z_0)}\right\|_{L^2(\mathbb{R}_+)} \leq Ch^2, \tag{V.5}$$

where the operators H_0^h and H_1^h are defined respectively by :

$$H_0^h = -h^2\partial_t^2 + \left(t - h^{1/2}\xi(\eta(z_0))\right)^2,$$

$$H_1^h = 2t\kappa_0\left(t - h^{1/2}\xi(\eta(z_0))\right)^2 - \kappa_0 t^2\left(t - h^{1/2}\xi(\eta(z_0))\right) + h^2\kappa_0\partial_t.$$

Let us denote by (cf. (II.24) and (II.25)) :

$$M_3\left(\frac{1}{2}, \gamma(z_0)\right) = M_3(\gamma(z_0)), \quad M_3(\alpha, \gamma(z_0)) = M_3 \text{ for } \alpha > \frac{1}{2}.$$

The next lemma permits to conclude an upper bound for the eigenvalue $\mu^{(1)}(\alpha, \gamma, h)$.

Lemma V.3 *Under the above notations, there exist positive constants C, h_0 such that, when $h \in]0, h_0]$, we have the following estimate :*

$$\left| q_{h,A,\Omega}^{\alpha,\gamma}(u_h) - \{\Theta(\eta(z_0)) - 2M_3(\alpha,\gamma(z_0))\kappa_0 h^{3/2}\} \|u_h\|^2_{L^2(\Omega)} \right| \leq C h^{\epsilon_\alpha},$$

where $\epsilon_\alpha = \inf(13/8, 2\alpha + \frac{1}{2})$ for $\alpha > \frac{1}{2}$ and $\epsilon_{1/2} = 13/8$.

Proof. Notice that in the support of u_h we have $\gamma(z) = \gamma(z_0) + \mathcal{O}(h^{1/8})$. Then this gives :

$$q_{h,A,\Omega}^{\alpha,\gamma}(u_h) - q_{h,A,\Omega}^{\alpha,\gamma(z_0)}(u_h) = \mathcal{O}(h^{9/8+\alpha}).$$

In view of Lemmas V.1 and V.2, we get the following estimate :

$$\left| q_{h,A,\Omega}^{\alpha,\gamma}(u_h) - \int_{\mathbb{R}_+} \left(H_1^h + H_0^h\right)\left(U^h \varphi_{\eta(z_0)}\right) \times U^h \varphi_{\eta(z_0)} dt \right| \leq C h^{13/8}. \tag{V.6}$$

We note also that we have the following relations :

$$(U^h)^\star H_0^h U^h = h\left\{-\partial_t^2 + (t - \xi(\eta(z_0)))^2\right\}, \tag{V.7}$$

$$(U^h)^\star H_1^h U^h = \kappa_0 h^{3/2} H_1, \tag{V.8}$$

where the operator H_1 is defined by :

$$H_1 = (t - \xi(\eta(z_0)))^3 - \xi(\eta(z_0))^2 (t - \xi(\eta(z_0))) + \partial_t.$$

By defining $K_3(\alpha, h) := \int_{\mathbb{R}_+} H_1 \varphi_{\eta(z_0)} \cdot \varphi_{\eta(z_0)} dt$, the estimate (V.6) reads as :

$$\left| q_{h,A,\Omega}^{\alpha,\gamma}(u_h) - \left\{h\Theta(\eta(z_0)) + K_3(\alpha,h)\kappa_0 h^{3/2}\right\} \right| \leq C h^{13/8}. \tag{V.9}$$

Now, for $\alpha = \frac{1}{2}$, we get by using (II.3) that $K_3(\frac{1}{2}, h) = -2M_3(\frac{1}{2}, \gamma(z_0))$. For $\alpha > \frac{1}{2}$, thanks to Propositions II.4 and II.5, we get that

$$K_3(\alpha, h) = -2M_3 + \mathcal{O}(h^{2\alpha - 1}).$$

Finally, the decay of $\varphi_{\eta(z_0)}$ in Proposition II.9 gives that $\|u_h\|_{L^2(\Omega)}$ is exponentially close to 1. This achieves the proof of the lemma. $\square$

The min-max principle gives now, thanks to Lemma V.3, an upper bound for $\mu^{(1)}(\alpha,\gamma,h)$. Under the hypothesis of Theorem I.4, we take z_0 such that

$$(\kappa_{\mathrm{r}}-3\gamma)(z_0)=(\kappa_{\mathrm{r}}-3\gamma)_{\max}$$

and we use the expansion (cf. (II.27)) :

$$\Theta(\eta(z_0))=\Theta_0+6M_3\gamma(z_0)h^{1/2}+\mathcal{O}(h).$$

Therefore, (V.9) gives the following upper bound :

$$\mu^{(1)}(1,\gamma,h)\leq h\Theta_0-2M_3(\kappa_{\mathrm{r}}-3\gamma)_{\max}h^{3/2}+\mathcal{O}(h^{13/8}). \tag{V.10}$$

Under the hypothesis of Theorem I.5, we choose z_0 such that $\kappa_{\mathrm{r}}(z_0)=(\kappa_{\mathrm{r}})_{\max}$.

B. Lower bound

As in the proof of Proposition III.3, we consider a standard scaled partition of unity[31] $\left(\chi_{j,h^{1/6}}\right)_{j\in\mathbb{Z}^2}$ of $\mathbb{R}^2$ that satisfies :

$$\sum_{j\in J}|\chi_{j,h^{1/6}}(z)|^2=1,\quad \sum_{j\in J}|\nabla\chi_{j,h^{1/6}}(z)|^2\leq Ch^{-1/3}, \tag{V.11}$$

$$\mathrm{supp}\ \chi_{j,h^{1/6}}\subset jh^{1/6}+[-h^{1/6},h^{1/6}]^2. \tag{V.12}$$

We define the following set of indices :

$$J^1_{\tau(h)}:=\{j\in\mathbb{Z}^2;\mathrm{supp}\chi_{j,h^{1/6}}\cap\Omega\neq\phi,\mathrm{dist}(\mathrm{supp}\ \chi_{j,h^{1/6}},\partial\Omega)\leq\tau(h)\},$$

where the number $\tau(h)$ is defined by :

$$\tau(h)=h^{\delta},\quad \text{with}\quad \frac{1}{6}\leq\delta\leq\frac{1}{2}, \tag{V.13}$$

and the number δ will be chosen in a suitable manner.

We consider also another scaled partition of unity in $\mathbb{R}$:

$$\psi^2_{0,\tau(h)}(t)+\psi^2_{1,\tau(h)}(t)=1,\quad |\psi'_{j,\tau(h)}(t)|\leq\frac{C}{\tau(h)},\quad j=0,1, \tag{V.14}$$

$$\operatorname{supp}\psi_{0,\tau(h)}\subset[\frac{\tau(h)}{20},+\infty[,\quad \operatorname{supp}\psi_{1,\tau(h)}\subset]-\infty,\frac{\tau(h)}{10}]. \tag{V.15}$$

Note that, for each $j\in J^1_{\tau(h)}$, the function $\psi_{1,\tau(h)}(t)\chi_{j,h^{1/6}}(s,t)$ could be interpreted, by means of boundary coordinates, as a function in $\overline{\Omega}$. Moreover, each $\psi_{1,\tau(h)}(t)\chi_{j,h^{1/6}}(s,t)$ is supported in a rectangle

$$K(j,h)=]-h^{1/6}+s_j,s_j+h^{1/6}[\times[0,h^\delta[$$

near $\partial\Omega$. The role of δ is then to control the size of the width of each rectangle $K(j,h)$. Due to the exponential decay of a ground state away from the boundary (Theorem IV.1), we get the following lemma.

Lemma V.4 *Suppose that $\alpha>\frac{1}{2}$. With the above notations, a L^2-normalized ground state $u_{\alpha,\gamma,h}$ of the operator $P^{\alpha,\gamma}_{h,A,\Omega}$ satisfies :*

$$\left|\sum_{j\in J^1_{\tau(h)}} q^{\alpha,\gamma}_{h,A,\Omega}\left(\chi_{j,h^{1/6}}\psi_{1,\tau(h)}u_{\gamma,h}\right)-\mu^{(1)}(\alpha,\gamma,h)\right|\leq Ch^{5/3}. \tag{V.16}$$

The proof of (V.16) follows the same lines of that in Ref. 10 (Formulas (10.4), (10.5) and (10.6)).

For each $j\in J^1_{\tau(h)}$, we define a unique point $z_j\in\partial\Omega$ by the relation $s(z_j)=s_j$. We denote then by $\kappa_j=\kappa_{\mathrm{r}}(z_j)$, $a_j(t)=1-\kappa_j t$, $A^j(t)=-t\left(1-\frac{t}{2}\kappa_j\right)$, and $\gamma_j=\gamma(z_j)$.

We consider now the k-family of one dimensional differential operators :

$$H_{h,j,k}=-h^2a_j^{-1}\partial_t(a_j\partial_t)+(1+2\kappa_j t)(hk-A^j)^2, \tag{V.17}$$

where k is a real parameter. We denote by $H^{\alpha,\gamma,D}_{h,j,k}$ the self-adjoint realization on $L^2\left(]0,h^\delta[;a_j(t)dt\right)$ of $H_{h,j,k}$ whose domain is given by :

$$D(H^{\gamma,D}_{h,j,k})=\{v\in H^2(]0,h^\delta[);v'(0)=h^\alpha\tilde{\gamma}_j v(0),\,v(h^\delta)=0\}. \tag{V.18}$$

The parameter $\tilde{\gamma}_j$ is defined by :

$$\tilde{\gamma}_j=\gamma_j+\varepsilon(h),$$

where $\varepsilon(h) = 0$ if the function γ is constant; if γ is not constant, then there are constants $C, h_0 > 0$ such that :

$$|\varepsilon(h)| \leq Ch^{1/6}, \quad \forall h \in]0, h_0].$$

We now introduce :

$$\mu_1^j(\alpha, \gamma, h) := \inf_{k\in\mathbb{R}} \inf \mathrm{Sp}(H_{h,j,k}^{\gamma,D}). \tag{V.19}$$

We have now the following lemma.

Lemma V.5 *For each $\alpha \in [\frac{1}{2}, +\infty[$, we have under the above notations :*

$$\mu^{(1)}(\alpha, \gamma, h) \geq \left(\inf_{j \in J^1_{\tau(h)}} \mu_1^j(\alpha, \gamma, h) \right) + \mathcal{O}(h^{5/3}). \tag{V.20}$$

Again the proof follows the same lines of Ref. 10 (Section 11), but let us explain briefly the main steps. We express each term $q_{h,A,\Omega}^{\alpha,\gamma}(\psi_{1,\tau(h)}\chi_{j,h^{1/6}}u_{\alpha,\gamma,h})$ in boundary coordinates. We work with the local choice of gauge given in Proposition A.2. We expand now all terms by Taylor's formula near $(s_j, 0)$. After controlling the remainder terms, thanks to the exponential decay of the ground states away from the boundary, we apply a partial Fourier transformation in the tangential variable s and we get finally the result of the lemma.

We have now to find, uniformly over $k \in \mathbb{R}$, a lower bound for the first eigenvalue $\mu_1^j(k; \alpha, \gamma, h)$ of the operator $H_{h,j,k}^{\alpha,\gamma,D}$. Putting $\beta = \kappa_j$, $\xi = -h^{1/2}k$ and $\eta = \tilde{\gamma}_j$, we get by a scaling argument :

$$\mu_1^j(k; \alpha, \gamma, h) = h\mu_1(H_{h,\beta,\xi}^{\alpha,\eta,D}),$$

where $\mu_1(H_{h,\beta,\xi}^{\alpha,\eta,D})$ is the first eigenvalue of the one dimensional operator :

$$\begin{aligned} H_{h,\beta,\xi}^{\alpha,\eta,D} &= -\partial_t^2 + (t-\xi)^2 + \beta h^{1/2}(1-\beta h^{1/2}t)^{-1}\partial_t \\ &\quad + 2\beta h^{1/2} t \left(t - \xi - \beta h^{1/2}\frac{t^2}{2} \right)^2 - \beta h^{1/2} t^2 (t-\xi) + \beta^2 h \frac{t^4}{4}, \end{aligned} \tag{V.21}$$

whose domain is defined by :

$$D(H_{h,\beta,\xi}^{\alpha,\eta,D}) = \{u \in H^2(]0, h^{\delta-1/2}[); u'(0) = h^{\alpha-1/2}\eta\, u(0),\ u(h^{\delta-1/2}) = 0\}.$$

We have then to find (when $\eta, \beta \in]-M, M[$ and M a given positive constant), uniformly with respect to $\xi \in \mathbb{R}$, a lower bound for the eigenvalue $\mu_1(H^{\alpha,\eta,D}_{h,\beta,\xi})$. The min-max principle gives the following preliminary localization of the spectrum of the operator $H^{\alpha,\eta,D}_{h,\beta,\xi}$:

Lemma V.6 *For each $M > 0$ and $\alpha \in [\frac{1}{2}, +\infty[$, there exist positive constants C, h_0 such that,*

$$\forall \eta, \beta \in]-M, M[, \quad \forall \xi \in \mathbb{R}, \quad \forall h \in]0, h_0],$$

we have,

$$\left| \mu_j(H^{\alpha,\eta,D}_{h,\beta,\xi}) - \mu_j(H^{\alpha,\eta,D}_{0,\xi}) \right| \leq C h^{2\delta - 1/2} \left(1 + \mu_j(H^{\alpha,\eta,D}_{0,\xi}) \right), \tag{V.22}$$

where, for an operator T having a compact resolvent, $\mu_j(T)$ denotes the increasing sequence of eigenvalues of T.

Remark V.7 *Note that the min-max principle gives now that*

$$\mu_j(H^{\alpha,\eta,D}_{0,\xi}) \geq \mu^{(j)}(h^{\alpha-1/2}\eta, \xi),$$

where, for $\tilde{\eta} \in \mathbb{R}$, $\mu^{(j)}(\tilde{\eta}, \xi)$ is the increasing sequence of eigenvalues of the operator $H[\eta, \xi]$ introduced in (II.2).

The following lemma deals with the case when ξ is not localized very close to $\xi(h^{\alpha-1/2}\eta)$.

Lemma V.8 *Suppose that $\delta \in]1/4, 1/2[$. For each $\alpha \geq \frac{1}{2}$, there exists $\rho \in]0, \delta - \frac{1}{4}]$, and for each $M > 0$, there exist positive constants $\zeta, h_0 > 0$ such that,*

$$\forall \eta, \beta \in]-M, M[, \quad \forall \xi \textit{ such that } |\xi - \xi(h^{\alpha-1/2}\eta)| \geq \zeta h^{\rho}, \quad \forall h \in]0, h_0],$$

we have,

$$\mu_1(H^{\alpha,\eta,D}_{h,\beta,\xi}) \geq \Theta(h^{\alpha-1/2}\eta) + h^{2\rho}. \tag{V.23}$$

Proof. It is sufficient to obtain (V.23) for $\mu^{(1)}(h^{\alpha-1/2}\eta, \xi)$, thanks to Lemma V.6 and Remark V.7. We start with the case when $\alpha = \frac{1}{2}$ and $\eta \in]-M, M[$. Writing Taylors

formula up to the second order for the function $\xi \mapsto \mu^{(1)}(\eta,\xi)$, we get positive constants θ, C_1 such that when $|\xi-\xi(\eta)|\le\theta$, we have :

$$\mu^{(1)}(\eta,\xi) \geq \Theta(\eta) + C_1|\xi-\xi(\eta)|^2.$$

Then by taking ζ such that $C_1\zeta > \zeta_0$, where $\zeta_0>1$ is a constant to be chosen appropriately, we get when $\zeta h^\rho \le |\xi-\xi(\eta)|\le\theta$,

$$\mu^{(1)}(\eta,\xi) \geq \Theta(\eta) + \zeta_0 h^{2\rho},$$

where ρ is also a positive constant to be chosen later. When $|\xi-\xi(\eta)|>\theta$, we get a positive constant ϵ_θ such that :

$$\mu^{(1)}(\eta,\xi) \geq \Theta(\eta) + \epsilon_\theta.$$

Then by choosing h_0 such that $\zeta_0 h_0^\rho < \epsilon_\theta$, we get for $|\xi-\xi(\eta)|\ge\zeta h^\rho$ and $h\in]0,h_0]$:

$$\mu^{(1)}(\eta,\xi) \geq \Theta(\eta) + \zeta_0 h^{2\rho}. \tag{V.24}$$

We treat now the case when $\alpha>1/2$. Note that the min-max principle gives uniformly for all $\xi\in\mathbb{R}$ and $\eta\in]-M,M[$,

$$\mu^{(1)}(h^{\alpha-1/2}\eta,\xi) \geq (1-C\eta_- h^{\alpha-1/2})\mu^{(1)}(0,\xi).$$

Then using (V.24) for $\eta=0$ and $\rho=\inf(\delta-\frac{1}{4},\alpha-\frac{1}{2})$, we can choose ζ_0 large enough so that we have for $|\xi-\xi_0|\ge\zeta h^\rho$:

$$\mu^{(1)}(h^{\alpha-1/2}\eta,\xi) \geq \Theta(h^{\alpha-1/2}\eta) + \frac{\zeta_0}{2}h^\rho.$$

To finish the proof, we replace ξ_0 by $\xi(h^{\alpha-1/2}\eta)$ getting an error of order $\mathcal{O}(h^{\alpha-1/2})$. $\square$

Now we deal with the case when $|\xi-\xi(h^{\alpha-1/2}\eta)|<\zeta h^\rho$. Let $\tilde\eta = h^{\alpha-1/2}\eta$. We look for a formal solution $\left(\mu, f^{\alpha,\eta}_{h,\beta,\xi}\right)$ of the spectral problem

$$H^{\alpha,\eta}_{h,\beta,\xi} f^{\alpha,\eta}_{h,\beta,\xi} = \mu f^{\alpha,\eta}_{h,\beta,\xi}, \quad \left(f^{\alpha,\eta}_{h,\beta,\xi}\right)'(0) = h^{\alpha-1/2} f^{\alpha,\eta}_{h,\beta,\xi}(0), \tag{V.25}$$

in the form :

$$\mu = d_0 + d_1\left(\xi - \xi(\tilde\eta)\right) + d_2\left(\xi - \xi(\tilde\eta)\right)^2 + d_3 h^{1/2}, \tag{V.26}$$

$$f^{\alpha,\eta}_{h,\beta,\xi} = u_0 + \left(\xi - \xi(\tilde\eta)\right)u_1 + \left(\xi - \xi(\tilde\eta)\right)^2 u_2 + h^{1/2}u_3, \tag{V.27}$$

where the coefficients d_0, d_1, d_2, d_3 and the functions u_0, u_1, u_2, u_3 are to be determined. We expand the operator $H^{\alpha,\eta,D}_{h,\beta,\xi}$ in powers of $(\xi - \xi(\tilde\eta))$ and then we identify the coefficients of the terms of orders $(\xi - \xi(\tilde\eta))^j$ $(j = 0, 1, 2)$ and $h^{1/2}$. We then obtain for the coefficients :

$$\begin{cases} d_0 = \Theta(\tilde\eta), \quad u_0 = \varphi_{\tilde\eta} \\ \\ d_1 = 0, \quad u_1 = 2\tilde R[\tilde\eta]\left\{(t - \xi(\tilde\eta))\varphi_{\tilde\eta}\right\} \\ \\ d_2 =: d_2(\alpha,\eta) = 1 - 2\int_{\mathbb{R}_+}(t - \xi(\tilde\eta))\varphi_{\tilde\eta}u_1 dt, \\ \\ u_2 = \tilde R[\tilde\eta]\left\{4\left(t - \xi(\tilde\eta)\right)\tilde R[\tilde\eta]\left[\left(t - \xi(\tilde\eta)\right)\varphi_{\tilde\eta}\right] - d_2\right\} \\ \\ d_3 =: d_3(\alpha,\eta) = \beta\int_{\mathbb{R}_+}\varphi_{\tilde\eta}\left\{\partial_t + (t - \xi(\tilde\eta))^3\right\}\varphi_{\tilde\eta}dt, \\ \\ u_3 = -\tilde R[\tilde\eta]\left[\beta\left(\partial_t + \left(t - \xi(\tilde\eta)\right)^3 - \xi(\tilde\eta)^2\left(t - \xi(\tilde\eta)\right)\right) - d_3\right]u_0. \end{cases} \tag{V.28}$$

Using the function $\chi(\frac{t}{h^{\delta-1/2}})f^{\alpha,\eta}_{h,\beta,\xi}$ (where χ is the same as in (III.5)) as a quasi-mode, we get by the spectral theorem, thanks to the decay results in Propositions II.9 and II.10 and to the localization of the spectrum in Lemma V.6, the following lemma.

Lemma V.9 *Suppose that* $\delta \in]\frac14, \frac12[$. *For each* $M > 0$ *and* $\alpha \in [\frac12, 1]$, *there exist positive constants* $C > 0, h_0$ *such that,*

$$\forall \eta, \beta \in]-M, M[, \quad \forall \xi \text{ such that } |\xi - \xi(\tilde\eta)| \leq \zeta h^\rho, \quad \forall h \in]0, h_0],$$

we have,

$$\begin{aligned} &\left|\mu_1(H^{\alpha,\eta,D}_{h,\beta,\xi}) - \left\{\Theta(\tilde\eta) + d_2(\alpha,\eta)(\xi - \xi(\tilde\eta))^2 + d_3(\alpha,\eta)h^{1/2}\right\}\right| \\ &\leq C\left[h^{1/2}|\xi - \xi(\tilde\eta)| + h^{\delta+1/2}\right]. \end{aligned} \tag{V.29}$$

where $d_2(\alpha,\eta)$ and $d_3(\alpha,\eta)$ are defined by (V.28) respectively.

Hence we have obtained by this analysis a lower bound for the first eigenvalue $\mu^{(1)}(\alpha,\gamma,h)$. We complete the picture by showing that the term $d_2(\alpha,\eta)$ is positive.

Lemma V.10 *For each $\alpha \in [\frac{1}{2},+\infty[$ and $M>0$, there exists a positive constant h_0 such that :*

$$d_2(\alpha,\eta) > 0, \quad \forall h \in]0,h_0], \quad \forall \eta \in]-M,M[.$$

Proof. It is actually sufficient to prove the conclusion of the lemma when $\alpha = \frac{1}{2}$. If $\alpha > \frac{1}{2}$, we replace $d_2(\alpha,\eta)$ by its approximation up to the first order, thanks to Proposition II.4, and we obtain that :

$$d_2(\alpha,\eta) = d_2(\frac{1}{2},0) + \mathcal{O}(h^{\alpha-1/2}),$$

which gives the lemma.

For the particular case $\alpha = \frac{1}{2}$, we show that :

$$d_2(\frac{1}{2},\eta) = \frac{1}{2}\left(\partial_\xi^2 \mu^{(1)}(\eta,\cdot)\right)(\xi(\eta))$$

which is strictly positive. □

We are now able to conclude the asymptotics given in Theorems I.4 and I.5. First we choose $\delta = \frac{5}{12}$. When $\alpha > \frac{1}{2}$ we replace $\Theta(\tilde{\eta})$ and d_3 by their approximations up to the second and first orders respectively, thanks to Propositions II.5 and II.4. For $\alpha = \frac{1}{2}$ we get by (II.21) that d_3 is indeed equal to $-2M_3(\frac{1}{2},\gamma)$.

Remark V.11 *When $\alpha \in]\frac{1}{2},1[$ and when the function γ is not constant, we get from the above analysis that the upper bound in Remark III.2 is indeed an asymptotic expansion, and we achieve therefore the proof of Theorem I.2.*

We get also that the quadratic form $q_{h,A,\Omega}^{\alpha,\gamma}$ can be bounded from below by means of a potential W :

$$q_{h,A,\Omega}^{\alpha,\gamma}(u) \geq \int_\Omega W(x)|u(x)|^2 dx, \quad \forall u \in H^1(\Omega),$$

where W is defined for some positive constant C_0 by :

$$W(x) = \begin{cases} h & ; \; if \;\; \mathrm{dist}(x, \partial\Omega) > h^{1/6} \\ h\Theta_0 + 6M_3\gamma(x)h^{\alpha+1/2} - C_0 h^{\inf(3/2,2\alpha)} & ; \; if \;\; \mathrm{dist}(x, \partial\Omega) < h^{1/6}. \end{cases}$$

Then, as in Section IV, we get by Agmon's technique that a ground state decays exponentially away from the boundary points where γ is minimum and hence we have completed the proof of Theorem I.3.

Remark V.12 *Note also that the above analysis permits, under the hypotheses of Theorems I.4 and I.5, to bound the quadratic form $q_{h,A,\Omega}^{\alpha,\gamma}$ from below using a potential W defined either by means of the function $\kappa_{\mathrm{r}} - 3\gamma$ (when $\alpha = 1$) or by the scalar curvature κ_{r} (when γ is constant). Then, by using Agmon's technique, we finish the proofs of Theorems I.4 and I.5.*

VI. CONCLUSION

The systematic analysis in the spirit of Ref. 10 has allowed us to understand the role of the boundary condition imposed by De Gennes. We have extended in Theorems I.4 and I.5 the expansion announced by Pan[25] in the particular case when $\alpha = 1$ and γ is a positive constant. However, there is a specific difficulty when γ is negative. We have not been able to obtain the localization of the ground state when $\alpha < 1/2$ and $\gamma_0 > 0$. This is strongly related to the question of the localization of the ground state of the Dirichlet realization of the Schrödinger operator with constant magnetic field which is open. Finally, in the spirit of Refs. 12,26,27, we hope to apply this analysis to the onset of superconductivity and to complete the analysis of Ref. 28 (cf. Ref. 18).

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APPENDIX A: COORDINATES NEAR THE BOUNDARY

We recall in this appendix well-known coordinates that straightens a portion of the boundary $\partial\Omega$. Let $s \in]-\frac{|\partial\Omega|}{2}, \frac{|\partial\Omega|}{2}] \mapsto M(s) \in \partial\Omega$ be a regular parametrization of $\partial\Omega$. For each $x \in \Omega$ and $\epsilon > 0$ we denote by :

$$t(x) = \operatorname{dist}(x, \partial\Omega) \text{ and } \Omega_\epsilon = \{x \in \overline{\Omega}; \operatorname{dist}(x, \partial\Omega) < \epsilon\}.$$

Then there exist a positive constant $t_0 > 0$ depending on Ω such that, for each $x \in \Omega_{t_0}$, we can define the coordinates $(s(x), t(x))$ by :

$$t(x) = |x - M(s(x))|,$$

and such that the transformation :

$$\psi : \Omega_{t_0} \ni x \mapsto (s(x), t(x)) \in \mathbb{S}^1_{|\partial\Omega|/2\pi} \times [0, t_0[$$

is a diffeomorphisim. The Jacobian of this coordinate transformation is given by :

$$a(s,t) = \det(D\psi) = 1 - t\kappa_{\mathrm{r}}(s). \tag{A.1}$$

To a vector field $A = (A_1, A_2) \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$, we associate the vector field $\tilde{A} = (\tilde{A}_1, \tilde{A}_2) \in C^\infty(\mathbb{S}^1_{|\partial\Omega|/2\pi} \times [0, t_0[)$ by the following relation :

$$\tilde{A}_1 ds + \tilde{A}_2 dt = A_1 dx_1 + A_2 dx_2. \tag{A.2}$$

We get then the following change of variable formulas.

Proposition A.1 *Let $u \in H^1(\Omega)$ be supported in Ω_{t_0}. Then we have :*

$$\int_{\Omega_{t_0}} |(h\nabla - iA)u|^2\,dx = \int_{\mathbb{S}^1_{|\partial\Omega|/2\pi}\times[0,t_0[} \left[|(h\partial_t - i\tilde{A}_2)v|^2 + a^{-2}|(h\partial_s - i\tilde{A}_1)v|^2\right] a\,dsdt. \quad \text{(A.3)}$$

and

$$\int_{\Omega_{t_0}} |u(x)|^2\,dx = \int_{\mathbb{S}^1_{|\partial\Omega|/2\pi}\times[0,\epsilon_0[} |v(s,t)|^2 a\,dsdt, \quad \text{(A.4)}$$

where $v(s,t) = u(\psi^{-1}(s,t))$.

We have also the relation :

$$(\partial_{x_1}A_2 - \partial_{x_2}A_1)\,dx_1 \wedge dx_2 = \left(\partial_s\tilde{A}_2 - \partial_t\tilde{A}_1\right) a^{-1} ds \wedge dt,$$

which gives,

$$\operatorname{curl}\tilde{A} = (1 - t\kappa_{\mathrm{r}}(s))\operatorname{curl}A.$$

We give in the next proposition a standard choice of gauge.

Proposition A.2 *Consider a vector field $A = (A_1, A_2) \in C^\infty(\overline{\Omega};\mathbb{R}^2)$ such that* curl$A = 1$. *For each point $x_0 \in \partial\Omega$, there exist a neighborhood $\mathcal{V}_{x_0} \subset \Omega_{t_0}$ of x_0 and a smooth real-valued function ϕ_{x_0} such the vector field $A_{new} := A - \nabla\phi_{x_0}$ satisfies :*

$$\tilde{A}^1_{new} = -t\left(1 - \frac{t}{2}\kappa_{\mathrm{r}}(s)\right) \text{ and } \tilde{A}^2_{new} = 0 \text{ in } \mathcal{V}_{x_0}, \quad \text{(A.5)}$$

with $\tilde{A}_{new} = (\tilde{A}^1_{new}, \tilde{A}^2_{new})$.

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[29] Actually we shall need this decay only when $\alpha < 1/2$ and $\gamma_0 < 0$.

[30] Actually, if z_0 is a point of maximum of κ_{r}, the remainder is better and of order $\mathcal{O}(h^{3/4})$ for the first term.

[31] We take a partition of unity associated to squares instead of discs.